Steger — Le sostituzioni trigonometriche

Dalle identità $\operatorname{sen}^2 + \cos^2 = 1$ e $\operatorname{sen}^2 / \cos^2 + \cos^2 / \cos^2 = 1/\cos^2$, cioè $\operatorname{tg}^2 + 1 = \sec^2$, si ricavano:

$$A^2 - A^2 \operatorname{sen}^2 = A^2 \cos^2 \qquad A^2 \operatorname{tg}^2 + A^2 = A^2 \sec^2 \qquad A^2 \sec^2 - A^2 = A^2 \operatorname{tg}^2$$

Sulla base di queste si trovano le 3 sostituzioni trigonometriche:

$$\begin{array}{rlrlrl}
x &= A \operatorname{sen} t & x &= A \operatorname{tg} t & x &= A \sec t \\
\sqrt{A^2 - x^2} &= A \cos t & \sqrt{x^2 + A^2} &= A \sec t & \sqrt{x^2 - A^2} &= A \operatorname{tg} t \\
dx &= A \cos t \, dt & dx &= A \sec^2 t \, dt & dx &= A \operatorname{tg} t \sec t \, dt \\
\multicolumn{2}{c}{-\pi/2 \le t \le \pi/2} & \multicolumn{2}{c}{-\pi/2 < t < \pi/2} & \multicolumn{2}{c}{0 \le t < \pi/2}
\end{array}$$

La sostituzione della prima colonna si usa se l'integranda contiene $\sqrt{A^2 - x^2}$, e può servire anche se l'integranda contiene $A^2 - x^2$. Idem per le altre due.

Nel esempio seguente si usa la sostituzione della seconda colonna: $x = A \operatorname{tg} t$.

$$\begin{aligned}
\int x^2 \sqrt{x^2 + A^2}\, dx &= \int A^2 \operatorname{tg}^2 t \, A \sec t \, A \sec^2 t \, dt = A^4 \int \operatorname{tg}^2 t \sec^3 t \, dt \\
&= A^4 \int (\sec^2 t - 1) \sec^3 t \, dt = A^4 \int \sec^5 t - \sec^3 t \, dt \\
&= A^4 \big(\frac{1}{4} \operatorname{tg} t \sec^3 t + \frac{3}{4} \cdot \frac{1}{2} \operatorname{tg} t \sec t + \frac{3}{4} \cdot \frac{1}{2} \log(\operatorname{tg} t + \sec t) - \frac{1}{2} \operatorname{tg} t \sec t - \frac{1}{2} \log(\operatorname{tg} t + \sec t) \big) \\
&= A^4 \big(\frac{1}{4} \operatorname{tg} t \sec^3 t - \frac{1}{8} \operatorname{tg} t \sec t - \frac{1}{8} \log(\operatorname{tg} t + \sec t) \big) \\
&= A^4 \left(\frac{1}{4} \frac{x}{A} \frac{(x^2 + A^2)^{3/2}}{A^3} - \frac{1}{8} \frac{x}{A} \frac{\sqrt{x^2 + A^2}}{A} - \frac{1}{8} \log(\frac{x}{A} + \frac{\sqrt{x^2 + A^2}}{A}) \right) \\
&= \frac{x(x^2 + A^2)^{3/2}}{4} - \frac{A^2 x \sqrt{x^2 + A^2}}{8} - \frac{A^4}{8} \log \left(\frac{x + \sqrt{x^2 + A^2}}{A} \right) \\
&= \frac{x(x^2 + A^2)^{3/2}}{4} - \frac{A^2 x \sqrt{x^2 + A^2}}{8} - \frac{A^4}{8} \log(x + \sqrt{x^2 + A^2}) + \frac{A^4 \log A}{8}
\end{aligned}$$

Si osserva che il termine $(A^4 \log A)/8$ è costante, quindi si potrebbe ommetterlo e ciò che rimarebbe sarebbe ancora una antiderivata di $x^2 \sqrt{x^2 + A^2}$.

1. Calcolare la derivata della risposta e controllare che vale $x^2 \sqrt{x^2 + A^2}$.

2. Sono da svolgere i calcoli indicati dai punti di sospensione $(\cdots)$.

$$\int \sqrt{x^2+A^2}\,dx = \cdots = \frac{x\sqrt{x^2+A^2}}{2} + \frac{A^2}{2}\log(x+\sqrt{x^2+A^2})$$

$$\int x^2+A^2\,dx = \cdots = \frac{x(x^2+A^2)}{3} + \frac{2A^2x}{3} = \cdots = \frac{x^3}{3} + A^2x$$

$$\int (x^2+A^2)^{3/2}\,dx = \cdots = \frac{x(x^2+A^2)^{3/2}}{4} + \frac{3A^2x\sqrt{x^2+A^2}}{8} + \frac{3A^4}{8}\log(x+\sqrt{x^2+A^2})$$

$$\begin{aligned}\int (x^2+A^2)^2\,dx &= \cdots = \frac{x(x^2+A^2)^2}{5} + \frac{4A^2x(x^2+A^2)}{15} + \frac{8A^4x}{15}\\ &= \cdots = \frac{x^5}{5} + \frac{2A^2x^3}{3} + A^4x\end{aligned}$$

$$\begin{aligned}\int (x^2+A^2)^{5/2} &= \cdots = \frac{x(x^2+A^2)^{5/2}}{6} + \frac{5A^2x(x^2+A^2)^{3/2}}{24}\\ &\qquad + \frac{5A^4x\sqrt{x^2+A^2}}{16} + \frac{5A^6}{16}\log(x+\sqrt{x^2+A^2})\end{aligned}$$

3.

$$\int \frac{1}{\sqrt{x^2+A^2}}\,dx = \cdots = \log(x+\sqrt{x^2+A^2})$$

$$\int \frac{1}{x^2+A^2}\,dx = \cdots = \frac{1}{A}\operatorname{arctg}(x/A)$$

$$\int \frac{1}{(x^2+A^2)^{3/2}}\,dx = \cdots = \frac{1}{A^2}\int \cos t\,dt = \cdots = \frac{x}{A^2\sqrt{x^2+A^2}}$$

$$\int \frac{1}{(x^2+A^2)^2}\,dx = \cdots = \frac{x}{2A^2(x^2+A^2)} + \frac{1}{2A^3}\operatorname{arctg}(x/A)$$

$$\int \frac{1}{(x^2+A^2)^{5/2}}\,dx = \cdots = \frac{x}{3A^2(x^2+A^2)^{3/2}} + \frac{2x}{3A^4\sqrt{x^2+A^2}}$$

4. Per questo esercizio e quello dopo si usa la sostituzione della terza colonna: $x = A\sec t$.

$$\begin{aligned}\int \sqrt{x^2-A^2}\,dx &= \cdots = A^2\int(\sec^2 t-1)\sec t\,dt\\ &= \cdots = \frac{x\sqrt{x^2-A^2}}{2} - \frac{A^2}{2}\log(x+\sqrt{x^2-A^2})\end{aligned}$$

$$\begin{aligned}\int (x^2-A^2)^{3/2}\,dx &= \cdots = A^4\int(\sec^2 t-1)^2\sec t\,dt\\ &= \cdots = \frac{x(x^2-A^2)^{3/2}}{4} - \frac{3A^2x\sqrt{x^2-A^2}}{8} + \frac{3A^4}{8}\log(x+\sqrt{x^2-A^2})\end{aligned}$$

5.

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-A^2}}\,dx &= \cdots = \log(x+\sqrt{x^2-A^2}) \\
\int \frac{1}{x^2-A^2}\,dx &= \cdots = \frac{1}{A}\int \csc t\,dt = \cdots = \frac{-1}{A}\log\left(\frac{x+A}{\sqrt{x^2-A^2}}\right) \\
&= \cdots = \frac{1}{2A}\log\left(\frac{x-A}{x+A}\right) \\
\int \frac{1}{(x^2-A^2)^{3/2}}\,dx &= \cdots = \frac{1}{A^2}\int \frac{\cos t}{\operatorname{sen}^2 t}\,dt = \frac{-1}{A^2\operatorname{sen} t} = \cdots = \frac{-x}{A^2\sqrt{x^2-A^2}} \\
\int \frac{1}{(x^2-A^2)^2}\,dx &= \cdots = \frac{1}{A^3}\int \frac{\cos^2 t}{\operatorname{sen}^3 t}\,dt = \ldots \frac{1}{A^3}\int \csc^3 t - \csc t\,dt \\
&= \cdots = \frac{-x}{2A^2(x^2-A^2)} - \frac{1}{4A^3}\log\left(\frac{x-A}{x+A}\right) \\
\int \frac{1}{(x^2-A^2)^{5/2}}\,dx &= \cdots = \frac{1}{A^4}\int \frac{\cos t}{\operatorname{sen}^4 t} - \frac{\cos t}{\operatorname{sen}^2 t}\,dt = \frac{1}{A^4}\left(\frac{-1}{3\operatorname{sen}^3 t} + \frac{1}{\operatorname{sen} t}\right) \\
&= \cdots = \frac{-x^3}{3A^4(x^2-A^2)^{3/2}} + \frac{x}{A^4\sqrt{x^2-A^2}} \\
&= \cdots = \frac{-x}{3A^2(x^2-A^2)^{3/2}} + \frac{2x}{3A^4\sqrt{x^2-A^2}}
\end{aligned}
$$

È interessante notare gli stretti paralleli fra le antiderivate calcolate in esercizi (**2**) e (**3**) e quelle calcolate in esercizi (**4**) e (**5**).

6. Per questo esercizio e quello dopo si usa la sostituzione della prima colonna: $x = A\operatorname{sen} t$.

$$
\begin{aligned}
\int \sqrt{A^2-x^2}\,dx &= \cdots = \frac{x\sqrt{A^2-x^2}}{2} + \frac{A^2\operatorname{arcsen}(x/A)}{2} \\
\int A^2-x^2\,dx &= \cdots = \frac{x(A^2-x^2)}{3} + \frac{2A^2x}{3} = \cdots = A^2x - \frac{x^3}{3} \\
\int (A^2-x^2)^{3/2}\,dx &= \cdots = \frac{x(A^2-x^2)^{3/2}}{4} + \frac{3A^2x\sqrt{A^2-x^2}}{8} + \frac{3A^2}{8}\operatorname{arcsen}(x/A) \\
\int (A^2-x^2)^2\,dx &= \cdots = \frac{x(A^2-x^2)^2}{5} + \frac{4A^2x(A^2-x^2)}{15} + \frac{8x}{15} \\
&= \cdots = A^4x - \frac{2A^2x^3}{3} + \frac{x^5}{5} \\
\int (A^2-x^2)^{5/2}\,dx &= \cdots = \frac{x(A^2-x^2)^{5/2}}{6} + \frac{5A^2x(A^2-x^2)^{3/2}}{24} \\
&\qquad + \frac{5A^4x\sqrt{A^2-x^2}}{16} + \frac{5A^6\operatorname{arcsen}(x/A)}{16}
\end{aligned}
$$

7.

$$\int \frac{1}{\sqrt{A^2-x^2}}\,dx = \cdots = \operatorname{arcsen}(x/A)$$
$$\int \frac{1}{A^2-x^2}\,dx = \cdots = \frac{1}{A}\log\left(\frac{x+A}{\sqrt{A^2-x^2}}\right) = \cdots = \frac{1}{2A}\log\left(\frac{A+x}{A-x}\right)$$
$$\int \frac{1}{(A^2-x^2)^{3/2}}\,dx = \cdots = \frac{x}{A^2\sqrt{A^2-x^2}}$$
$$\int \frac{1}{(A^2-x^2)^2}\,dx = \cdots = \frac{x}{2A^2(A^2-x^2)} + \frac{1}{4A^3}\log\left(\frac{A+x}{A-x}\right)$$
$$\int \frac{1}{(A^2-x^2)^{5/2}}\,dx = \cdots = \frac{x}{3A^2(A^2-x^2)^{3/2}} + \frac{2x}{3A^4\sqrt{A^2-x^2}}$$

Per gli integrali (con limiti), spesso ci sono notevoli semplificazioni. Nel esempio seguente, usiamo la sostituzione $x = A\operatorname{sen} t$, quindi $x = \pm A$ corrispondono a $A\operatorname{sen} t = \pm A$, cioè $\operatorname{sen} t = \pm 1$, cioè $t = \pm\pi/2$.

$$\int_{x=-A}^{x=+A} x^4\sqrt{A^2-x^2}\,dx = \int_{t=-\pi/2}^{t=+\pi/2} (A\operatorname{sen} t)^4 (A\cos t)(A\cos t\,dt)$$
$$= A^6\int_{t=-\pi/2}^{t=+\pi/2} \operatorname{sen}^4 t\cos^2 t\,dt = A^6\int_{t=-\pi/2}^{t=+\pi/2} \operatorname{sen}^4 t - \operatorname{sen}^6 t\,dt$$

Poi

$$\int_{t=-\pi/2}^{t=+\pi/2} \operatorname{sen}^4 t\,dt = \frac{-1}{4}\cos t\operatorname{sen}^3 t\Big|_{t=-\pi/2}^{t=+\pi/2} + \frac{3}{4}\int_{t=-\pi/2}^{t=+\pi/2}\operatorname{sen}^2 t\,dt$$
$$= 0 + \frac{3}{4}\cdot\frac{-1}{2}\cos t\operatorname{sen} t\Big|_{t=-\pi/2}^{t=+\pi/2} + \frac{3}{4}\cdot\frac{1}{2}\int_{t=-\pi/2}^{t=+\pi/2} 1\,dt$$
$$= 0 + 0 + \frac{3\pi}{8} = \frac{3\pi}{8}$$
$$\int_{t=-\pi/2}^{t=+\pi/2} \operatorname{sen}^6 t\,dt = \frac{-1}{6}\cos t\operatorname{sen}^5 t\Big|_{t=-\pi/2}^{t=+\pi/2} + \frac{5}{6}\int_{t=-\pi/2}^{t=+\pi/2}\operatorname{sen}^4 t\,dt$$
$$= 0 + \frac{5}{6}\cdot\frac{3\pi}{8} = \frac{5\pi}{16}$$

onde

$$\int_{x=-A}^{x=+A} x^4\sqrt{A^2-x^2}\,dx = A^6\left(\frac{3\pi}{8} - \frac{5\pi}{16}\right) = \frac{\pi A^6}{16}$$

8.

$$\int_{x=-A}^{x=+A} x^2\sqrt{A^2-x^2}\,dx = \cdots = \pi A^4/8$$