DA SCANNERIZZARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 22 dicembre 2023

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 13.00 alle ore 13.30. La prova si concluderà puntualmente.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

Le formule per le coordinate sferiche sono:

$$x = r\operatorname{sen}\theta\cos\phi \qquad y = r\operatorname{sen}\theta\operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\operatorname{sen}\theta\,dr\,d\theta\,d\phi$$

A. Calcolare:

$$\int \frac{8t^3+6t^2+4t+2}{(1-t^2)^2}\,dt$$

B. Sia $R > 0$ un parametro fisso. Sia $\mathcal{D}'$ il disco determinato dalla condizione:

$$(x-R)^2 + y^2 \le R^2$$

Prima, fare uno schizzo di $\mathcal{D}'$. Poi, usare le coordinate polari e calcolare la media di x^3 su $\mathcal{D}'$.

C. Sia $R > 0$ un parametro fisso. Sia $\mathcal{O}$ l'ottava parte della sfera determinata da:

$$x, y, z \geq 0 \qquad\qquad x^2 + y^2 + z^2 \leq R^2$$

Usare le coordinate sferiche e calcolare:

$$\iiint_{\mathcal{O}} z^3 \, dV \qquad\qquad \iiint_{\mathcal{O}} x^3 \, dV \qquad\qquad \iiint_{\mathcal{O}} y^3 \, dV$$

e verificare che le tre risposte sono uguali.

D. Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio delle soluzioni del sistema:

$$\begin{aligned} p + q &= A \\ p + q + r &= B \\ q + r + s &= C \\ r + s + t &= D \\ s + t &= E \end{aligned}$$

per p, q, r, s, e t in termini di A, B, C, D, e E. Si scoprirà che non c'è nessuna soluzione se A, B, C, D, e E non soddisfano una certa condizione.

1. Specificare la condizione su A, B, C, D, e E.

2. Dare la parametrizzazione.

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$\boxed{A}$ $$\int \frac{8t^3+6t^2+4t+2}{(1-t^2)^2}\,dt$$

grado = 3

grado = 4

3 < 4

$$(1-t^2)^2 = [(1+t)(1-t)]^2 = (1+t)^2(1-t)^2$$

$$\frac{8t^3+6t^2+4t+2}{(1+t)^2(1-t)^2} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{C}{1-t} + \frac{D}{(1-t)^2}$$

$$8t^3+6t^2+4t+2$$
$$= A(1+t)(1-t)^2 + B(1-t)^2 + C(1-t)(1+t)^2 + D(1+t)^2$$

$t=1$ $\quad 8+6+4+2 = 0+0+0+D(1+1)^2$

$\hookrightarrow 20 = 4D \rightarrow D = 5$

$t=-1$ $\quad -8+6-4+2 = 0+B(1-(-1))^2+0+0$

$\hookrightarrow -4 = 4B \rightarrow B = -1$

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$$8t^3+6t^2+4t+2$$

$$= \left(\begin{aligned} &A(1-t^2)(1-t) \\ &+B(1-t)^2 \\ &+C(1-t^2)(1+t) \\ &+D(1+t)^2 \end{aligned} \right) = \left(\begin{aligned} &A(t^3-t^2-t+1) \\ &+-1(\quad t^2-2t+1) \\ &+C(-t^3-t^2+t+1) \\ &+5(\quad t^2+2t+1) \end{aligned} \right)$$

$$\rightsquigarrow \left\{ \begin{aligned} A-C&=8 \\ -A-1-C+5&=6 \\ -A+2+C+10&=4 \\ A-1+C+5&=2 \end{aligned} \right\} \longrightarrow \left\{ \begin{aligned} A-C&=8 \\ -A-C&=2 \\ -A+C&=-8 \\ A+C&=-2 \end{aligned} \right\}$$

$$\rightsquigarrow \left\{ \begin{aligned} A-C&=8 \\ A+C&=-2 \end{aligned} \right\} \rightsquigarrow \left\{ \begin{aligned} 2A&=(A-C)+(A+C)=6 \\ 2C&=(A+C)-(A-C)=-10 \end{aligned} \right\}$$

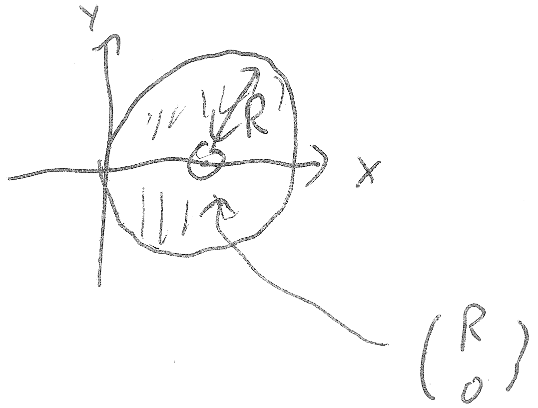
$$\rightsquigarrow A=3, \quad C=-5$$

$$\int 3\cdot\frac{1}{1+t} - \frac{1}{(1+t)^2} - 5\,\frac{1}{1-t} + 5\,\frac{1}{(1-t)^2}\,dt$$

$$= 3\log(1+t) + \frac{1}{1+t} + 5\log(1-t) + 5\,\frac{1}{1-t}$$

$\boxed{B}$

$$(x-R)^2+y^2 = \mathrm{dist}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} R \\ 0 \end{pmatrix}\right)^2$$



$$(x-R)^2+y^2 \le R^2$$

$$\hookrightarrow \quad x^2-2Rx+R^2+y^2 \le R^2$$

$$\hookrightarrow \quad x^2+y^2 \le 2Rx$$

$$\hookrightarrow \quad r^2 \le 2Rr\cos\theta$$

$$\hookrightarrow \quad r \le 2R\cos\theta$$

Dal disegno, o perché occorre $\cos\theta \ge 0$, $-\pi/2 \le \theta \le \pi/2$

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$$\iint_{D'} 1 \, dA = \int_{\theta=-\pi/2}^{\pi/2} \left(\int_{r=0}^{2R\cos\theta} 1 \cdot r \, dr \right) d\theta$$

$$= \frac{4R^2}{2} \int_{\theta=-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$\Big[\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^0\theta \, d\theta$$

$$= \frac{1}{2}\pi \Big]$$

$$= 2R^2 \cdot \frac{\pi}{2} = \pi R^2 \quad (\text{giusto})$$

$$\iint_{D} x^3 \, dA = \int_{\theta=-\pi/2}^{\pi/2} \left(\int_{r=0}^{2R\cos\theta} (r\cos\theta)^3 \cdot r \, dr \right) d\theta$$

$$= \frac{(2R)^5}{5} \int_{\theta=-\pi/2}^{\pi/2} \cos^3\theta \cdot \cos^5\theta \, d\theta$$

$$= \frac{32R^5}{5} \int_{\theta=-\pi/2}^{\pi/2} \cos^8\theta \, d\theta$$

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$$\left[\int_{-\pi/2}^{\pi/2} \cos^8\theta \, d\theta = \frac{7}{8} \int_{-\pi/2}^{\pi/2} \cos^6\theta \, d\theta \right.$$

$$= \frac{7}{8} \cdot \frac{5}{6} \int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$\left. = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \pi \right]$$

$$= \frac{32 R^5}{5} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi$$

$$= R^5 \cdot 7 \cdot \frac{1}{6} \cdot 3 \cdot \frac{1}{2} \pi = \frac{7 \pi R^5}{4}$$

$$\overline{X^3} = \frac{7 \pi R^5 / 4}{\pi R^2} = \frac{7}{4} R^3$$

$\boxed{C}$

raggio = R

Si ricordi che $\rho \geq 0$, $0 \leq \theta \leq \pi$, $\operatorname{sen}\theta \geq 0$, $-\pi \leq \varphi \leq \pi$ si usano sempre per le coordinate sferiche.

$$x^2+y^2+z^2 \leq R^2 \to \rho^2 \leq R^2 \to 0 \leq \rho \leq R$$

$$z \geq 0 \to \rho \cos\theta \geq 0 \to \cos\theta \geq 0$$

$$\to 0 \leq \theta \leq \frac{\pi}{2}$$

$$\begin{matrix} x \geq 0 \\ y \geq 0 \end{matrix} \to \begin{matrix} \rho \operatorname{sen}\theta \cos\varphi \geq 0 \\ \rho \operatorname{sen}\theta \operatorname{sen}\varphi \geq 0 \end{matrix} \to \begin{matrix} \cos\varphi \geq 0 \\ \operatorname{sen}\varphi \geq 0 \end{matrix}$$

$$\to 0 \leq \varphi \leq \frac{\pi}{2}$$

$$\iiint_{\sigma} z^3 \, dV = \int_{\varphi=0}^{\pi/2} \left(\int_{\theta=0}^{\pi/2} \left(\int_{\rho=0}^{R} (\rho\cos\theta)^3 \cdot \rho^2 \operatorname{sen}\theta \, d\rho \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \left(\int_{\theta=0}^{\pi/2} \cos^3\theta \operatorname{sen}\theta \left(\int_{\rho=0}^{R} \rho^5 \, d\rho \right) d\theta \right) d\varphi$$

$$= \frac{\pi}{2} \cdot \frac{R^6}{6} \int_{\theta=0}^{\pi/2} \cos^3\theta \operatorname{sen}\theta \, d\theta$$

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$$= \frac{\pi R^6}{12}\left[-\frac{\cos^4\theta}{4}\right]_{\theta=0}^{\pi/2} = \frac{\pi R^6}{48}(0-(-1))$$

$$= \frac{\pi R^6}{48}$$

$$\iiint_{\theta} x^3 \, dV$$

$$= \int_{\varphi=0}^{\pi/2}\left(\int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{R}(r \operatorname{sen}\theta \cos\varphi)^3 r^2 \operatorname{sen}\theta \, dr\right)d\theta\right)d\varphi$$

$$= \int_{\varphi=0}^{\pi/2}\cos^3\varphi\left(\int_{\theta=0}^{\pi/2}\operatorname{sen}^4\theta\left(\int_{r=0}^{R} r^5 \, dr\right)d\theta\right)d\varphi$$

$$\left[\ \int_{\varphi=0}^{\pi/2}\cos^3\varphi \, d\varphi = \frac{2}{3}\int_{\varphi=0}^{\pi/2}\cos^1\varphi \, d\varphi = \frac{2}{3}\ \right]$$

$$\left[\ \int_{\theta=0}^{\pi/2}\operatorname{sen}^4\theta \, d\theta = \frac{3}{4}\int_{\theta=0}^{\pi/2}\operatorname{sen}^2\theta \, d\theta\right.$$

$$= \frac{3}{4}\cdot\frac{1}{2}\int_{\theta=0}^{\pi/2}\operatorname{sen}^0\theta \, d\theta = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$\left.= \frac{3\pi}{16}\ \right]$$

$$= \int_{\varphi=0}^{\pi/2} \cos^3\varphi \left(\int_{\theta=0}^{\pi/2} \operatorname{sen}^4\theta \cdot \frac{R^6}{6}\, d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \cos^3\varphi \cdot \frac{3\pi}{16} \cdot \frac{R^6}{6}\, d\varphi$$

$$= \frac{2}{3} \cdot \frac{3}{16} \cdot \frac{\pi R^6}{6} = \frac{\pi R^6}{48}$$

$\iiint_{\Theta} y^3 \, dV$ è quasi lo stesso, come calcolo; esattamente lo stesso come esito.

$\boxed{D}$

p	q	r	s	t	=	A	B	C	D	E
1	1	0	0	0		1	0	0	0	0
1	1	1	0	0		0	1	0	0	0
0	1	1	1	0		0	0	1	0	0
0	0	1	1	1		0	0	0	1	0
0	0	0	1	1		0	0	0	0	1

$$E_2 \leftarrow E_2 - E_1 \quad \longrightarrow \quad \begin{matrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$E_2 \leftrightarrow E_3 \quad \longrightarrow \quad \begin{matrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$E_1 \leftarrow E_1 - E_2 \quad \longrightarrow \quad \begin{matrix} 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

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$E1 \leftarrow E1 + E3$
$E2 \leftarrow E2 - E3$
$\longrightarrow$
$E4 \leftarrow E4 - E3$

$$\begin{array}{ccccc|ccccc}
1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$E1 \leftarrow E1 + E4$
$E2 \leftarrow E2 - E4$
$\longrightarrow$
$E5 \leftarrow E5 - E4$

$$\begin{array}{ccccc|ccccc}
p & q & m & s & t = & A & B & C & D & E \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & +1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1
\end{array}$$

$p + t = A - C + D$

$q - t = C - D$

$m = -A + B$

$s + t = A - B + D$

$0 = -A + B - D + E$

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(1) Soluzioni esistono solo se

$$-A+B-D+E=0$$

(ossia $B+E=A+D$)

(2)
$$\begin{aligned}
p &= -t + A - C + D\\
q &= t + C - D\\
r &= -A + B\\
s &= -t + A - B + D\\
t &= t
\end{aligned}$$

Controllo

$$p+q = \begin{aligned}&(-t+A-C+D)\\ &+(t+C-D) = A \checkmark\end{aligned}$$

$$p+q+r = \begin{aligned}&(-t+A-C+D)\\ &+(t+C-D)\\ &+(-A+B) = B \checkmark\end{aligned}$$

$$q+r+s = \begin{aligned}&(t+C-D)\\ &+(-A+B)\\ &+(-t+A-B+D) = C \checkmark\end{aligned}$$

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$$
\begin{aligned}
A+S+C &= (-A+B) \\
&\quad + (-C + A - B + D) \\
&\quad + (C) = D \quad Ⓥ
\end{aligned}
$$

$$
\begin{aligned}
S+C &= (-C + A - B + D) \\
&\quad + (C) = A - B + D \\
&= E \quad Ⓥ
\end{aligned}
$$

perché $-A + B - D + E = 0$