DA FOTOCOPIARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 21 giugno 2022

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.
I compiti corretti saranno a disposizione venerdì 24 giugno, alle 12.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

Le formule per le coordinate sferiche sono:

$$x = r\operatorname{sen}\theta\cos\phi \qquad y = r\operatorname{sen}\theta\operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\operatorname{sen}\theta\,dr\,d\theta\,d\phi$$

A. Usare la sostituzione $x = \operatorname{sen} t$ e calcolare

$$\int \frac{1}{\cos^3 t}\,dt$$

B. Sia $R > 0$ un parametro fisso, e sia $\mathcal{Z}$ la zona nel piano definito da

$$x^2 + y^2 \leq R^2 \qquad\qquad |y| \leq x$$

Fare uno schizzo di $\mathcal{Z}$. Poi usare le coordinate polari e calcolare la media di x sulla zona $\mathcal{Z}$.

C. Sia $R > 0$ e sia $\mathcal{S}_2$ la sfera di raggio R e punto centrale $\left(\begin{smallmatrix} 0 \\ R \\ 0 \end{smallmatrix}\right)$. Usare le coordinate sferiche e calcolare la media di y sulla sfera $\mathcal{S}_2$.

D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema:

$$\begin{array}{cccc} & y & +2z & = A \\ x & +2y & +3z & = B \\ x & +3y & +6z & = C \end{array}$$

per le incognite x, y, e z in termine dei parametri A, B, e C.

$\boxed{A}$

$$x = \operatorname{sen} t$$

$$dx = \cos t \; dt$$

$$\int \frac{1}{\cos^3 t} \, dt = \int \frac{1}{\cos^4 t} \cos t \, dt$$

$$= \int \frac{1}{(1 - \operatorname{sen}^2 t)^2} \cos t \, dt$$

$$= \int \frac{1}{(1 - x^2)^2} \, dx$$

$$= \int \frac{1}{(1-x)^2 (1+x)^2} \, dx$$

$$\frac{1}{(1-x)^2 (1+x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1+x)} + \frac{D}{(1+x)^2}$$

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$$1 = A(1-x)(1+x)^2 + B(1+x)^2$$
$$+ C(1-x)^2(1+x) + D(1-x)^2$$

$$x=1 \rightsquigarrow 1 = 0 + B(1+1)^2 + 0 + 0 = 4B$$
$$\rightsquigarrow B = 1/4$$

$$x=-1 \rightsquigarrow 1 = 0 + 0 + 0 + D(1-(-1))^2 = 4D$$
$$\rightsquigarrow D = 1/4$$

$$1 = A(1-x^2)(1+x) + B(1+2x+x^2)$$
$$+ C(1-x^2)(1-x) + D(1-2x+x^2)$$

$$= A(1+x-x^2-x^3)$$
$$+ B(1+2x+x^2)$$
$$+ C(1-x-x^2+x^3)$$
$$+ D(1-2x+x^2)$$

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$$\begin{cases} A+B+C+D=1 \\ A+2B-C-2D=0 \\ -A+B-C+D=0 \\ -A+C=0 \end{cases}$$

$$\begin{cases} A+C=1-B-D=1/2 \\ A-C=-2B+2D=0 \\ -A-C=-B-D=-1/2 \\ -A+C=0 \end{cases}$$

$$\begin{cases} A+C=1/2 \\ A=C \end{cases} \longrightarrow A=C=1/4$$

$$\int \frac{1}{(1-x)^2(1+x)^2}\,dx$$

$$=\int \frac{1}{4}\frac{1}{1-x}+\frac{1}{4}\frac{1}{(1+x)^2}+\frac{1}{4}\frac{1}{(1+x)}+\frac{1}{4}\frac{1}{(1+x)^2}\,dx$$

$$= -\frac{1}{4}\log(1-x) + \frac{1}{4}\frac{1}{1-x} + \frac{1}{4}\log(1+x) - \frac{1}{4}\frac{1}{1+x}$$

$$= -\frac{1}{4}\log(1-\operatorname{sen} t) + \frac{1}{4}\cdot\frac{1}{1-\operatorname{sen} t} + \frac{1}{4}\log(1+\operatorname{sen} t) - \frac{1}{4}\frac{1}{1+\operatorname{sen} t}$$

$$= \frac{1}{4}\log\left(\frac{1+\operatorname{sen} t}{1-\operatorname{sen} t}\right) + \frac{1}{4}\frac{(1+\operatorname{sen} t)-(1-\operatorname{sen} t)}{(1-\operatorname{sen} t)(1+\operatorname{sen} t)}$$

$$= \frac{1}{4}\log\left(\frac{(1+\operatorname{sen} t)^2}{(1-\operatorname{sen} t)(1+\operatorname{sen} t)}\right) + \frac{1}{4}\cdot\frac{2\operatorname{sen} t}{1-\operatorname{sen}^2 t}$$

$$= \frac{1}{4}\log\left(\frac{(1+\operatorname{sen} t)^2}{1-\operatorname{sen}^2 t}\right) + \frac{1}{2}\frac{\operatorname{sen} t}{\cos^2 t}$$

$$= \frac{1}{4}\log\left(\left(\frac{1+\operatorname{sen} t}{\cos t}\right)^2\right) + \frac{1}{2}\frac{\operatorname{sen} t}{\cos t}\cdot\frac{1}{\cos t}$$

$$= \frac{1}{2}\log\left(\frac{1+\operatorname{sen} t}{\cos t}\right) + \frac{1}{2}\operatorname{tg} t \cdot \sec t$$

$$= \frac{1}{2}\log(\operatorname{tg} t + \sec t) + \frac{1}{2}\operatorname{tg} t \cdot \sec t$$

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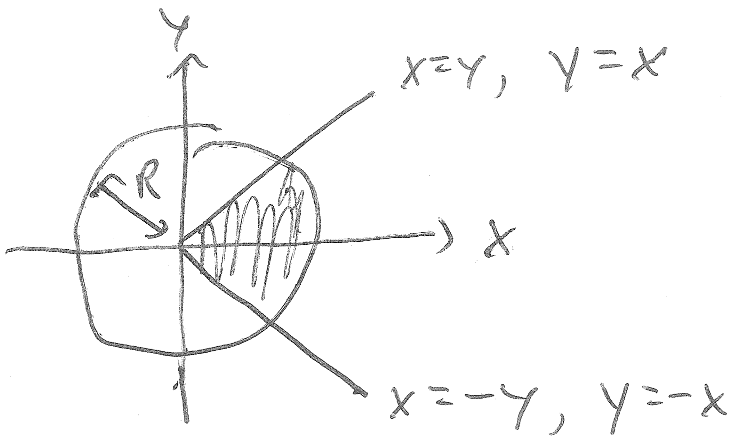
$\boxed{B}$



$$\iint_{\mathcal{Z}} 1\, dA = \int_{\theta=-\pi/4}^{\pi/4} \left(\int_{r=0}^{R} 1\cdot r\, dr \right) d\theta$$

$$= \int_{\theta=-\pi/4}^{\pi/4} \frac{R^2}{2}\, d\theta = \frac{\pi}{2}\cdot\frac{R^2}{2} = \frac{\pi R^2}{4}$$

$$\iint_{\mathcal{Z}} x\, dA = \int_{\theta=-\pi/4}^{\pi/4} \left(\int_{r=0}^{R} r\cos\theta \cdot r\, dr \right) d\theta$$

$$= \int_{\theta=-\pi/4}^{\pi/4} \cos\theta \cdot \frac{R^3}{3}\, d\theta$$

$$= \frac{R^3}{3}\left(+\operatorname{sen}\theta\right)\Big]_{\theta=-\pi/4}^{\pi/4} = \frac{R^3}{3}\left[\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{\sqrt{2}R^3}{3}$$

$$\bar{x} = \frac{\iint_{\mathcal{Z}} x \, dA}{\iint_{\mathcal{Z}} 1 \, dA} = \frac{\sqrt{2}R^3/3}{\pi R^2/4} = \frac{4\sqrt{2}}{3\pi} R$$

$\boxed{C}$ $dist\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ R \\ 0 \end{pmatrix}\right) \leq R$

$\hookrightarrow \sqrt{(x-0)^2 + (y-R)^2 + (z-0)^2} \leq R$

$\hookrightarrow x^2 + y^2 - 2Ry + R^2 + z^2 \leq R^2$

$\hookrightarrow x^2 + y^2 + z^2 \leq 2Ry$

$\hookrightarrow r^2 \leq 2R r \operatorname{sen}\theta \operatorname{sen}\varphi$

$\hookrightarrow r \leq 2R \operatorname{sen}\theta \operatorname{sen}\varphi$

$0 \leq r \leq 2R \operatorname{sen}\theta \operatorname{sen}\varphi$

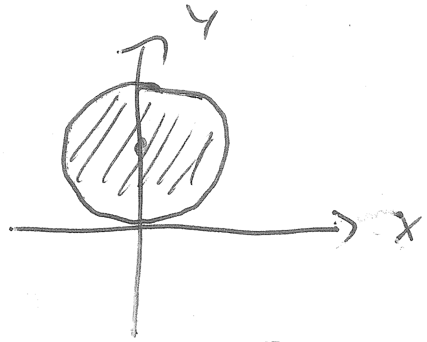
$\hookrightarrow 0 \leq 2R \operatorname{sen}\theta \operatorname{sen}\varphi$

$\hookrightarrow 0 \leq \operatorname{sen}\varphi$ (poiché $R \geq 0$, $\operatorname{sen}\theta \geq 0$)

$\hookrightarrow 0 \leq \varphi \leq \pi$

$0 \leq \theta \leq \pi$ (questo vincolo si usa sempre)

Proiezione sul piano-(x,y)



$$\iiint_{\mathcal{S}_2} 1 \, dV = \int_{\varphi=0}^{\pi} \left(\int_{\theta=0}^{\pi} \left(\int_{r=0}^{2R\operatorname{sen}\theta \operatorname{sen}\varphi} 1 \cdot r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi} \left(\int_{\theta=0}^{\pi} \frac{1}{3} 8R^3 \operatorname{sen}^3\theta \operatorname{sen}^3\varphi \operatorname{sen}\theta \right) d\theta \Big) d\varphi$$

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I.

$$\int_{t=0}^{\pi} 1\, dt = \pi$$

$$\int_{t=0}^{\pi} \operatorname{sen} t\, dt = -\cos t\Big]_{t=0}^{\pi} = -(-1)-(-1)$$

$$= 2$$

$$\int_{t=0}^{\pi} \operatorname{sen}^2 t\, dt = \frac{-1}{2} \operatorname{sen} t \cos t\Big]_{t=0}^{\pi}$$

$$+ \frac{1}{2} \int_{t=0}^{\pi} 1\, dt$$

$$= 0 + \frac{1}{2}\pi = \frac{1}{2}\pi$$

$$\int_{t=0}^{\pi} \operatorname{sen}^3 t\, dt = -\frac{1}{3} \cos t \operatorname{sen}^2 t\Big]_{t=0}^{\pi}$$

$$+ \frac{2}{3} \int_{t=0}^{\pi} \operatorname{sen} t\, dt$$

$$= 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$$

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$$\int_{t=0}^{\pi} \operatorname{sen}^4 t \, dt = -\frac{1}{4} \cos t \operatorname{sen}^3 t \Big]_{t=0}^{\pi} + \frac{3}{4} \int_{t=0}^{\pi} \operatorname{sen}^2 t \, dt$$

$$= 0 + \frac{3}{4} \cdot \frac{1}{2}\pi = \frac{3}{8}\pi$$

$$\int_{t=0}^{\pi} \operatorname{sen}^5 t \, dt = -\frac{1}{5} \cos t \operatorname{sen}^4 t \Big]_{t=0}^{\pi} + \frac{4}{5} \int_{t=0}^{\pi} \operatorname{sen}^3 t \, dt$$

$$= 0 + \frac{4}{5} \cdot \frac{4}{3} = \frac{16}{15}$$

$$\int_{t=0}^{\pi} \operatorname{sen}^6 t \, dt = -\frac{1}{6} \cos t \operatorname{sen}^5 t \Big]_{t=0}^{\pi} + \frac{5}{6} \int_{t=0}^{\pi} \operatorname{sen}^4 t \, dt$$

$$= 0 + \frac{5}{6} \cdot \frac{3}{8}\pi = \frac{5}{16}\pi$$

$$\iiint_{\mathcal{S}_2} 1 \, dV$$

$$= \frac{8R^3}{3} \int_{\varphi=0}^{\pi} \operatorname{sen}^3 \varphi \left(\int_{\theta=0}^{\pi} \operatorname{sen}^4 \theta \, d\theta \right) d\varphi$$

$$= \frac{8R^3}{3} \int_{\varphi=0}^{\pi} \operatorname{sen}^3 \varphi \cdot \frac{3}{8} \pi \cdot d\varphi$$

$$= \frac{8R^3}{3} \cdot \frac{4}{3} \cdot \frac{3}{8} \pi = \frac{4\pi R^3}{3}$$

$$\iiint_{\mathcal{S}_2} y \, dV$$

$$= \int_{\varphi=0}^{\pi} \left(\int_{\theta=0}^{\pi} \left(\int_{r=0}^{2R \operatorname{sen}\theta \operatorname{sen}\varphi} r \operatorname{sen}\theta \operatorname{sen}\varphi \cdot r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi} \left(\int_{\theta=0}^{\pi} \frac{1}{4} (2R \operatorname{sen}\theta \operatorname{sen}\varphi)^4 \operatorname{sen}\theta \operatorname{sen}\varphi \operatorname{sen}\theta \, d\theta \right) d\varphi$$

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$$= \frac{16}{4} R^4 \int_{\varphi=0}^{\pi} \operatorname{sen}^5 \varphi \left(\int_{\theta=0}^{\pi} \operatorname{sen}^6 \theta \, d\theta \right) d\varphi$$

$$= 4R^4 \int_{\varphi=0}^{\pi} \operatorname{sen}^5 \varphi \cdot \frac{5}{16} \pi$$

$$= 4R^4 \cdot \frac{16}{15} \cdot \frac{5}{16} \pi = \frac{4 \pi R^4}{3}$$

$$\overline{Y} = \frac{\iiint_{\mathcal{B}_2} y \, dV}{\iiint_{\mathcal{B}_2} 1 \, dV} = \frac{4\pi R^4/3}{4\pi R^3/3} = R$$

$\boxed{D}$

x	y	z = A	B	C
0	1	2	1	0	0
1	2	3	0	1	0
1	3	6	0	0	1

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$$\begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{E3 \leftarrow E3 - E1} \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[E3 \leftarrow E3 - E2]{E1 \leftarrow E1 - 2E2} \begin{pmatrix} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[E2 \leftarrow E2 - 2E3]{E1 \leftarrow E1 + E3} \begin{array}{cccccc} x & y & z = A & B & C \\ \hline 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}$$

$$x = -3A + C$$
$$y = 3A + 2B - 2C$$
$$z = -A - B + C$$

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Controllo:

$$y + 2z = 3A + 2B - 2C$$
$$+ 2(-A - B + C) = A \; ✓$$

$$x + 2y + 3z = -3A + C$$
$$+ 2(3A + 2B - 2C)$$
$$+ 3(-A - B + C) = B \; ✓$$

$$x + 3y + 6z = -3A + C$$
$$+ 3(3A + 2B - 2C)$$
$$+ 6(-A - B + C) = C \; ✓$$