DA FOTOCOPIARE

**Matematica 2, Laurea Triennale (Steger)**

**Prova Scritta di 9 settembre 2021**

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.
I compiti corretti saranno a disposizione lunedì 13 settembre, alle 12.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad dx\,dy = r\,dr\,d\theta$$

Le formule per le coordinate sferiche sono:

$$x = r\operatorname{sen}\theta\cos\phi \qquad y = r\operatorname{sen}\theta\operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\operatorname{sen}\theta\,dr\,d\theta\,d\phi$$

**A.** Calcolare

$$\int \frac{s}{s^2+8s+15}\,ds$$

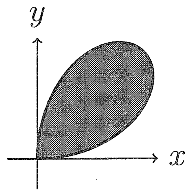
1. dopo l'applicazione della sostituzione $s = t - 4$, e
2. senza nessuna sostituzione,

e paragonare i due esiti.

**B.** Sia $\mathcal{Z}$ la zona del piano definita da:

$$x, y \geq 0 \qquad\qquad r^3 \leq xy \quad \text{dove } r = \sqrt{x^2 + y^2}$$

Usare le coordinate polari e calcolare l'area di $\mathcal{Z}$.




**C.** Sia $R > 0$ un parametro fisso e sia $\mathcal{S}$ la sfera definita da:

$$(x - R)^2 + y^2 + z^2 \leq R^2$$

Usare le coordinate sferiche e calcolare:

$$\iiint_{\mathcal{S}} r \, dV$$

dove $r = \sqrt{x^2 + y^2 + z^2}$.

**D.** Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio dei vettori:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

che sono perpendicolari a tutti e i 4 dei seguenti vettori:

$$\begin{pmatrix} 2 \\ -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

$\boxed{A}$ 1

$s = t-4$
$ds = dt$ $\rightsquigarrow \int \frac{s}{s^2+8s+15}\,ds$

$= \int \frac{t-4}{(t-4)^2+8(t-4)+15}\,dt$

$= \int \frac{t-4}{t^2-8t+16+8t-32+15}\,dt$

$= \int \frac{t-4}{t^2-1}\,dt$

$\frac{t-4}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1}$

$t-4 = A(t+1) + B(t-1)$

$t=+1 \rightsquigarrow 1-4 = A(1+1)+0 \rightarrow A = \frac{-3}{2}$

$t=-1 \rightsquigarrow -1-4 = 0 + B(-1-1) \rightsquigarrow B = \frac{5}{2}$

$$\int \frac{t-4}{t^2-1}\,dt = \int -\frac{3}{2}\,\frac{1}{t-1} + \frac{5}{2}\,\frac{1}{t+1}\,dt$$

$$= -\frac{3}{2}\log(t-1) + \frac{5}{2}\log(t+1)$$

$$= -\frac{3}{2}\log(s+4-1) + \frac{5}{2}\log(s+4+1)$$

$$= -\frac{3}{2}\log(s+3) + \frac{5}{2}\log(s+5)$$

② $$\int \frac{s}{s^2+8s+15}\,ds = \int \frac{s}{(s+3)(s+5)}\,ds$$

$$\frac{s}{(s+3)(s+5)} = \frac{C}{s+3} + \frac{D}{s+5}$$

$$\hookrightarrow\; s = C(s+5) + D(s+3)$$

$$s=-5 \rightsquigarrow -5 = 0 + D(-5+3) \rightsquigarrow D = \frac{5}{2}$$

$$s=-3 \rightsquigarrow -3 = C(-3+5) + 0 \rightsquigarrow C = -3/2$$

$$\int \frac{s}{(s+3)(s+5)}\,ds = \int -\frac{3}{2}\,\frac{1}{s+3} + \frac{5}{2}\,\frac{1}{s+5}\,ds$$

$$= -\frac{3}{2}\log(s+3) + \frac{5}{2}\log(s+5)$$

Le due risposte coincidono!

$\boxed{B}$ $x, y \geq 0 \rightsquigarrow r\cos\theta,\ r\,\mathrm{sen}\,\theta \geq 0$

$$\rightarrow \cos\theta,\ \mathrm{sen}\,\theta \geq 0$$

$$\rightsquigarrow 0 \leq \theta \leq \pi/2$$

$$r^3 \leq xy \rightsquigarrow r^3 \leq r(r\cos\theta)(r\,\mathrm{sen}\,\theta)$$

$$\rightsquigarrow r \leq \mathrm{sen}\,\theta\cos\theta$$

$$\iint_Z 1\,dA = \int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{\mathrm{sen}\,\theta\cos\theta} 1\cdot r\,dr\right)d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{2}(\mathrm{sen}\,\theta\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \operatorname{sen}^2\theta \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} (1-\cos^2\theta)\cos^2\theta \, d\theta$$

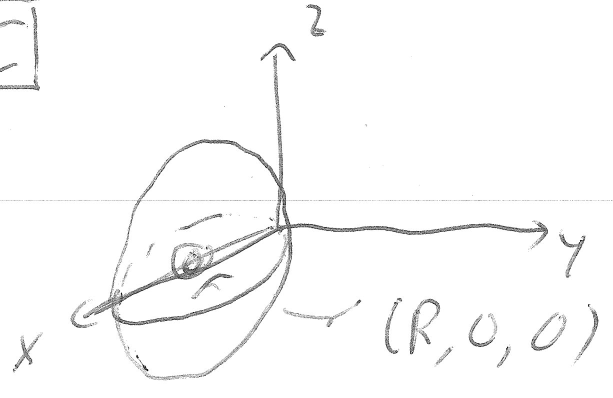
$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta - \cos^4\theta \, d\theta$$

$$\left[ \int_0^{\pi/2} \cos^2\theta \, d\theta = 0 + \frac{1}{2}\int_0^{\pi/2} \cos^0\theta \, d\theta = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4} \right.$$

$$\left. \int_0^{\pi/2} \cos^4\theta \, d\theta = 0 + \frac{3}{4}\int_0^{\pi/2} \cos^2\theta \, d\theta = \frac{3}{4}\cdot\frac{\pi}{4} = \frac{3\pi}{16} \right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{3\pi}{16}\right) = \frac{1}{2}\left(\frac{\pi}{16}\right) = \boxed{\frac{\pi}{32}}$$

$\boxed{C}$




$$(x-R)^2 + y^2 + z^2 \leq R^2$$

$$\longrightarrow x^2 - 2Rx + R^2 + y^2 + z^2 \leq R^2$$

$$\rightsquigarrow x^2 + y^2 + z^2 \leq 2Rx$$

$$\rightarrow \rho^2 \leq 2R(\rho \operatorname{sen}\theta \cos\varphi)$$

$$\rightsquigarrow \rho \leq 2R \operatorname{sen}\theta \cos\varphi$$

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$0 \leq r \leq 2R \operatorname{sen}\theta \cos\varphi$

Poi $0 \leq 2R \operatorname{sen}\theta \cos\varphi$

$\rightsquigarrow \cos\varphi \geq 0 \to -\pi/2 \leq \varphi \leq \pi/2$

in quanto $0 \leq \theta \leq \pi$ e $\operatorname{sen}\theta \geq 0$.

$$\iiint_S r \, dV$$

$$= \int_{\varphi=-\pi/2}^{\pi/2} \left( \int_{\theta=0}^{\pi} \left( \int_{r=0}^{2R \operatorname{sen}\theta \cos\varphi} r \cdot r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=-\pi/2}^{\pi/2} \left( \int_{\theta=0}^{\pi} \frac{1}{4} (2R \operatorname{sen}\theta \cos\varphi)^4 \operatorname{sen}\theta \, d\theta \right) d\varphi$$

$$= 4R^4 \int_{\varphi=-\pi/2}^{\pi/2} \cos^4\varphi \left( \int_{\theta=0}^{\pi} \operatorname{sen}^5\theta \, d\theta \right) d\varphi$$

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$$\left[\ \int_{\theta=0}^{\pi} \operatorname{sen}^5\theta \, d\theta = 0 + \frac{4}{5}\int_{\theta=0}^{\pi} \operatorname{sen}^3\theta \, d\theta\right.$$

$$= \frac{4}{5}\left(0 + \frac{2}{3}\int_{\theta=0}^{\pi} \operatorname{sen}\theta \, d\theta\right)$$

$$= \frac{4}{5}\cdot\frac{2}{3}\left(-\cos\theta\right)\Big]_{\theta=0}^{\pi} = \frac{4}{5}\cdot\frac{2}{3}\cdot 2 = \frac{16}{15}$$

$$\int_{\varphi=-\pi/2}^{\pi/2} \cos^4\varphi \, d\varphi = 0 + \frac{3}{4}\int_{-\pi/2}^{\pi/2} \cos^2\varphi \, d\varphi$$

$$= \frac{3}{4}\left(0 + \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^0\varphi \, d\varphi\right)$$

$$\left.= \frac{3}{4}\cdot\frac{1}{2}\cdot\pi = \frac{3\pi}{8}\ \right]$$

$$\iiint_B r \, dV = 4R^4 \int_{\varphi=-\pi/2}^{\pi/2} \cos^4\varphi \cdot \frac{16}{15}\, d\varphi$$

$$= 4R^4 \cdot \frac{3\pi}{8}\cdot\frac{16}{15} = 8R^4\pi/5$$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \text{ perpendicolare a } \begin{pmatrix} 2 \\ -2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\leftrightsquigarrow 0 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \circ \begin{pmatrix} 2 \\ -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} = 2x_1 - 2x_2 - 1x_3 + 0x_4 + 1x_5$$

è altrettanto per gli altri 3 vettori dati.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $= 0$ |
|---|---|---|---|---|---|
| 2 | -2 | -1 | 0 | 1 | |
| 1 | 2 | -2 | -1 | 0 | |
| 0 | 1 | 2 | -2 | -1 | |
| -1 | 0 | 1 | 2 | -2 | |

$E1 \leftarrow E1/2$

| | | | | |
|---|---|---|---|---|
| 1 | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| 1 | 2 | -2 | -1 | 0 |
| 0 | 1 | 2 | -2 | -1 |
| -1 | 0 | 1 | 2 | -2 |

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$$\xrightarrow[E4 \leftarrow E4 + E1]{E2 \leftarrow E2 - E1} \begin{matrix} 1 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 3 & -3/2 & -1 & -1/2 \\ 0 & 1 & 2 & -2 & -1 \\ 0 & -1 & 1/2 & 2 & -3/2 \end{matrix}$$

$$\xrightarrow{E2 \leftarrow \frac{1}{3} E2} \begin{matrix} 1 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 1 & 2 & -2 & -1 \\ 0 & -1 & 1/2 & 2 & -3/2 \end{matrix}$$

$$\begin{matrix} E1 \leftarrow E1 + E2 \\ E3 \leftarrow E3 - E2 \\ \longrightarrow \\ E4 \leftarrow E4 + E2 \end{matrix} \quad \begin{matrix} 1 & 0 & -1 & -1/3 & 1/3 \\ 0 & 1 & -1/2 & -1/3 & -1/6 \\ 0 & 0 & 5/2 & -5/3 & -5/6 \\ 0 & 0 & 0 & 5/3 & -5/3 \end{matrix}$$

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$$E3 \leftarrow \tfrac{2}{5} E3 \longrightarrow \begin{matrix} 1 & 0 & -1 & -1/3 & +1/3 \\ 0 & 1 & -1/2 & -1/3 & -1/6 \\ 0 & 0 & 1 & -2/3 & -1/3 \\ 0 & 0 & 0 & 5/3 & -5/3 \end{matrix}$$

$$\begin{matrix} E1 \leftarrow E1 + E3 \longrightarrow \\ E2 \leftarrow E2 + \tfrac{1}{2} E3 \end{matrix} \quad \begin{matrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2/3 & -1/3 \\ 0 & 0 & 1 & -2/3 & -1/3 \\ 0 & 0 & 0 & 5/3 & -5/3 \end{matrix}$$

$$E4 \leftarrow \tfrac{3}{5} E4 \longrightarrow \begin{matrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2/3 & -1/3 \\ 0 & 0 & 1 & -2/3 & -1/3 \\ 0 & 0 & 0 & 1 & -1 \end{matrix}$$

$E1 \leftarrow E1 + E4$

$E2 \leftarrow E2 + 2/3 \cdot E4$

$E3 \leftarrow E3 + 2/3 \cdot E4$

$$
\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & = 0 \\
\hline
1 & 0 & 0 & 0 & -1 & \\
0 & 1 & 0 & 0 & -1 & \\
0 & 0 & 1 & 0 & -1 & \\
0 & 0 & 0 & 1 & -1 &
\end{array}
$$

$$
\begin{aligned}
x_1 \qquad\qquad - x_5 &= 0 \\
x_2 \qquad - x_5 &= 0 \\
x_3 - x_5 &= 0 \\
x_4 - x_5 &= 0
\end{aligned}
\rightsquigarrow
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_5 \\ x_5 \\ x_5 \\ x_5 \\ x_5 \end{pmatrix}
$$

$$
= x_5 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
$$

<u>Controllo</u>: $2x_1 - 2x_2 - x_3 + x_5$

$= 2x_5 - 2x_5 - x_5 + x_5 = 0$

...