DA FOTOCOPIARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 26 febbraio 2020

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente. I compiti corretti saranno a disposizione venerdì 28 ottobre, alle 12.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate cilindriche sono:

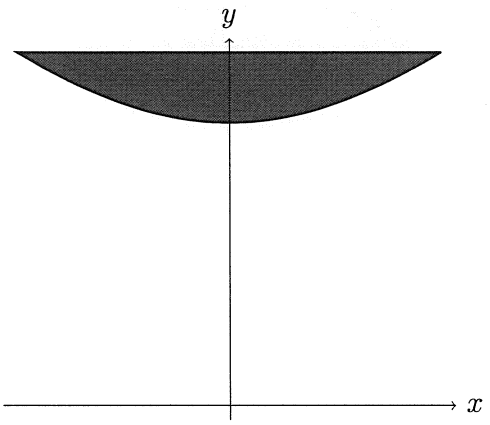
$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

A. Calcolare

$$\int \frac{z^2}{z^4 - 5z^2 + 4}\,dz$$

B. Usare le coordinate rettangolari (Cartesiane) e calcolare l'area della zona $\mathcal{Z}$ definita da

$$0 \leq y \leq 5 \qquad\qquad y^2 - x^2 \geq 16$$



C. Sia $R > 0$ un parametro fisso. Usare le coordinate cilindriche e calcolare la media di xyz sul volume $\mathcal{V}$ definita da

$$x, y, z \geq 0 \qquad\qquad x^2 + y^2 + z^2 \leq R^2$$

Si osserva che $\mathcal{V}$ è l'ottava parte di una sfera.

D. Trovare la soluzione del sistema:

$$\begin{array}{lll} p+q=A & p+q+r=B & q+r+s=C \\ r+s+t=D & s+t+u=E & t+u=F \end{array}$$

per le incognite p, q, r, s, t, u in termine dei parametri A, B, C, D, E, F.

A $z^4 - 5z^2 + 4$

$= (z^2 - 1)(z^2 - 4)$

$= (z-1)(z+1)(z-2)(z+2)$

$$\frac{z^2}{z^4 - 5z^2 + 4} = \frac{A}{z-1} + \frac{B}{z+1} + \frac{C}{z-2} + \frac{D}{z+2}$$

$$z^2 = A(z+1)(z-2)(z+2) + B(z-1)(z-2)(z+2) + C(z-1)(z+1)(z+2) + D(z-1)(z+1)(z-2)$$

$z = 1 \rightsquigarrow 1 = A(1+1)(1-2)(1+2) + 0 + 0 + 0$

$\rightsquigarrow 1 = -6A \rightsquigarrow A = -1/6$

$z = -1 \rightsquigarrow (-1)^2 = 0 + B(-2)(-3)(1) + 0 + 0$

$\rightsquigarrow 1 = 6B \rightsquigarrow B = 1/6$

$z=2 \rightsquigarrow (2)^2 = 0+0+C(1)(3)(4)+0$

$\rightsquigarrow 4 = 12C \rightsquigarrow C = 1/3$

$z=-2 \rightsquigarrow (-2)^2 = 0+0+0+D(-3)(-1)(-4)$

$\rightsquigarrow 4 = -12D \rightsquigarrow D = -1/3$

$$\int -\frac{1}{6}\frac{1}{z-1} + \frac{1}{6}\frac{1}{z+1} + \frac{1}{3}\frac{1}{z-2} - \frac{1}{3}\frac{1}{z+2}\,dz$$

$$= -\frac{1}{6}\log(z-1) + \frac{1}{6}\log(z+1) + \frac{1}{3}\log(z-2) - \frac{1}{3}\log(z+2)$$

B $y^2 - x^2 \geq 16 \rightsquigarrow y^2 \geq 16 + x^2$

$\rightsquigarrow y \geq \sqrt{16+x^2}$ (in quanto $y \geq 0$)

$$\sqrt{16+x^2} \le y \le 5$$

$$\sqrt{16+x^2} \le 5 \rightsquigarrow 16+x^2 \le 25$$

$$\rightsquigarrow x^2 \le 9 \rightsquigarrow -3 \le x \le 3$$

$$\int_{-3}^{3} \left(\int_{\sqrt{16+x^2}}^{5} 1 \, dy \right) dx$$

$$= \int_{-3}^{3} 5 - \sqrt{16+x^2} \, dx$$

$$= 5(3-(-3)) - \int_{-3}^{3} \sqrt{16+x^2} \, dx$$

$$= 30 - \int_{-3}^{3} \sqrt{16+x^2} \, dx$$

$x = 4 \operatorname{tg} \theta$

$dx = 4 \sec^2 \theta \, d\theta$

$\sqrt{16+x^2} = \sqrt{16+16\operatorname{tg}^2\theta} = \sqrt{16 \sec^2\theta} = 4 \sec\theta$

p. 4/14

$x = +3 \rightsquigarrow 4\,\mathrm{tg}\,\theta = +3 \rightsquigarrow \mathrm{tg}\,\theta = 3/4$

$\rightsquigarrow \theta = \mathrm{arctg}(3/4)$

$x = -3 \rightsquigarrow \cdots \rightsquigarrow \theta = -\mathrm{arctg}(3/4)$ ↑

$$= 30 - \int_{-\mathrm{arctg}(3/4)}^{+\mathrm{arctg}(3/4)} 4\sec\theta \cdot 4\sec^2\theta\, d\theta$$

$$= 30 - 16 \int_{-\mathrm{arctg}(3/4)}^{+\mathrm{arctg}(3/4)} \sec^3\theta\, d\theta$$

$$= 30 - 16 \cdot \frac{1}{2}\left(\mathrm{tg}\,\theta\sec\theta + \log(\mathrm{tg}\,\theta + \sec\theta)\right)\Big]_{-\mathrm{arctg}(\frac{3}{4})}^{+\mathrm{arctg}(\frac{3}{4})}$$

$\int$

$\mathrm{tg}(\mathrm{arctg}(\frac{3}{4})) = \frac{3}{4}$

$\sec(\mathrm{arctg}(\frac{3}{4})) = \sqrt{1 + \mathrm{tg}^2(\mathrm{arctg}\,\frac{3}{4})}$

$= \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

$\mathrm{tg}(\mathrm{arctg}(-\frac{3}{4})) = -\frac{3}{4}$

$\sec(\mathrm{arctg}(-\frac{3}{4})) = 5/4$ ↑

$$= 30 - 8\left(\frac{3}{4}\cdot\frac{5}{4} + \log\left(\frac{3}{4}+\frac{5}{4}\right) - \left(-\frac{3}{4}\right)\frac{5}{4} - \log\left(-\frac{3}{4}+\frac{5}{4}\right)\right)$$

$$= 30 - 8\left(\frac{15}{16} + \log(2) + \frac{15}{16} - \log\left(\frac{1}{2}\right)\right)$$

$$= 30 - 8\left(\frac{15}{8} + \log(2) + \log(2)\right)$$

$$= 30 - 15 - 16\log(2)$$

$$= \boxed{15 - 16\log(2)} \simeq 3{,}91$$

[C] $r^2 = x^2 + y^2$

$x^2 + y^2 + z^2 \leq R^2 \rightsquigarrow r^2 + z^2 \leq R^2$

$\rightsquigarrow z \leq \sqrt{R^2 - r^2}$

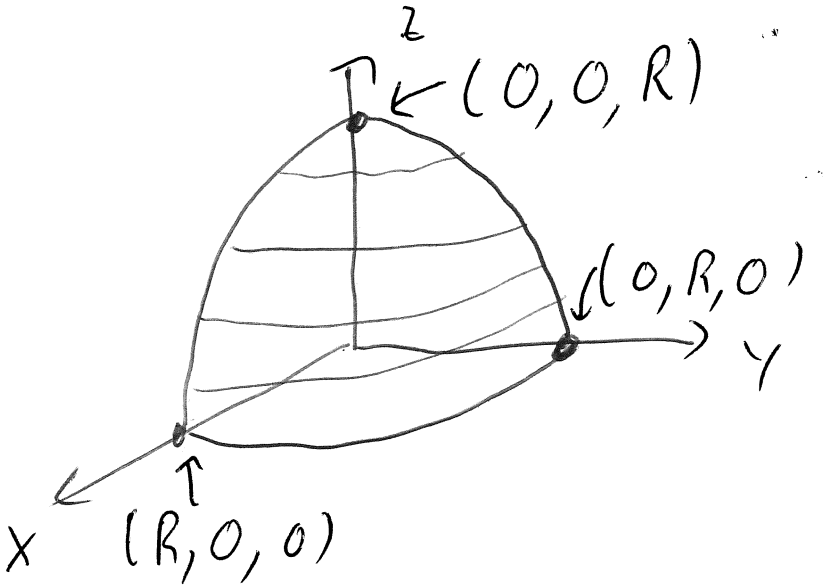
(in quanto $z \geq 0$)

$$0 \le z \le \sqrt{R^2 - r^2}$$

Questo ha senso solo se $R^2 - r^2 \ge 0$, quindi $r \le R$.

$$x, y \ge 0 \rightsquigarrow \begin{matrix} r\cos\theta \ge 0 \\ r\operatorname{sen}\theta \ge 0 \end{matrix} \rightsquigarrow \begin{matrix} \cos\theta \ge 0 \\ \operatorname{sen}\theta \ge 0 \end{matrix}$$

$$\rightsquigarrow 0 \le \theta \le \pi/2$$



$$\iiint_V 1 \, dV = \frac{1}{8}\left(\frac{4\pi R^3}{3}\right) = \frac{\pi R^3}{6}$$

poiché V è l'ottava parte della sfera.

$$\iiint_V xyz \, dV$$

$$= \int_{\theta=0}^{\pi/2} \left(\int_{m=0}^{R} \left(\int_{z=0}^{\sqrt{R^2-m^2}} (m \operatorname{sen}\theta)(m\cos\theta) z \cdot m \, dz \right) dm \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta\cos\theta \left(\int_{m=0}^{R} m^3 \left(\int_{z=0}^{\sqrt{R^2-m^2}} z \, dz \right) dm \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta\cos\theta \left(\int_{\mu=0}^{R} \mu^3 \frac{\left(\sqrt{R^2-\mu^2}\right)^2}{2} d\mu \right) d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta\cos\theta \left(\int_{\mu=0}^{R} \mu^3 R^2 - \mu^5 \, d\mu \right) d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta\cos\theta \left(\frac{R^6}{4} - \frac{R^6}{6} \right) d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{12} R^6 \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta\cos\theta \, d\theta$$

$$= \frac{R^6}{24} \cdot \frac{1}{2} \operatorname{sen}^2\theta \Big]_{\theta=0}^{\pi/2}$$

$$= \frac{R^6}{24} \cdot \left(\frac{1}{2} - 0 \right) = R^6/48$$

$$\overline{xyz} = \frac{R^6/48}{\pi R^3/6} = \boxed{\frac{R^3}{8\pi}}$$

[D] Metodo veloce:

$$\left.\begin{aligned} p+q+r &= B \\ p+q &= A \end{aligned}\right\} \quad r = B-A$$

$$\left.\begin{aligned} s+t+u &= E \\ t+u &= F \end{aligned}\right\} \quad s = E-F$$

$$q+r+s = C \rightsquigarrow q = C-r-s$$

$$\rightsquigarrow \boxed{q = A-B+C-E+F}$$

$$r+s+t = D \rightsquigarrow t = D-r-s$$

$$\rightsquigarrow \boxed{t = A-B+D-E+F}$$

$$p+q = A \rightsquigarrow p = A-q$$

$$= A-(A-B+C-E+F)$$

$$\rightsquigarrow \boxed{p = B-C+E-F}$$

$$t+u = F \rightsquigarrow u = F-t$$

$$= F-(A-B+D-E+F)$$

$$\rightsquigarrow \boxed{u = -A+B-D+E}$$

Gauss-Jordan :

p	q	m	s	t	u	=	A	B	C	D	E	F
①	1	0	0	0	0		1	0	0	0	0	0
	1	1	1	0	0	0		0	1	0	0	0
	0	1	1	1	0	0		0	0	1	0	0
	0	0	1	1	1	0		0	0	0	1	0
	0	0	0	1	1	1		0	0	0	0	1
	0	0	0	0	1	1		0	0	0	0	0

$E2 \leftarrow E2 - E1$ →

①	1	0	0	0	0		1	0	0	0	0	0
	0	⓪	1	0	0	0		-1	1	0	0	0
	0	1	1	1	0	0		0	0	1	0	0
	0	0	1	1	1	0		0	0	0	1	0
	0	0	0	1	1	1		0	0	0	0	1
	0	0	0	0	1	1		0	0	0	0	0

$E2 \leftrightarrow E3$ ⇝

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$$\begin{array}{cccccc|cccccc}
\textcircled{1} & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \textcircled{1} & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$E1 \leftarrow E1 - E2$

$$\begin{array}{cccccc|cccccc}
\textcircled{1} & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & \textcircled{1} & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \textcircled{1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$E1 \leftarrow E1 + E3$

$E2 \leftarrow E2 - E3$

$E4 \leftarrow E4 - E3$

$$\begin{array}{cccccc|cccccc}
\textcircled{1} & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & \textcircled{1} & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & \textcircled{1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \textcircled{1} & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$E1 \leftarrow E1+E4$

$E2 \leftarrow E2-E4$

$E5 \leftarrow E5-E4$

$$\begin{array}{cccccc|cccccc}
\textcircled{1} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & \textcircled{1} & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & \textcircled{1} & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \textcircled{1} & 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \underline{0} & 1 & -1 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$E5 \leftrightarrow E6$

p.13/14

$$\begin{matrix}
\boxed{1} & 0 & 0 & 0 & 1 & 0 \\
0 & \boxed{1} & 0 & 0 & -1 & 0 \\
0 & 0 & \boxed{1} & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{1} & 1 & 0 \\
0 & 0 & 0 & 0 & \boxed{1} & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{matrix}
\qquad
\begin{matrix}
1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & 0 & -1 & 1 & 0
\end{matrix}$$

$E1 \leftarrow E1 - E5$

$E2 \leftarrow E2 + E5$

$\rightsquigarrow$

$E4 \leftarrow E4 - E5$

$$\begin{matrix}
\boxed{1} & 0 & 0 & 0 & 0 & -1 \\
0 & \boxed{1} & 0 & 0 & 0 & 1 \\
0 & 0 & \boxed{1} & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{1} & 0 & -1 \\
0 & 0 & 0 & 0 & \boxed{1} & 1 \\
0 & 0 & 0 & 0 & 0 & \boxed{1}
\end{matrix}
\qquad
\begin{matrix}
1 & 0 & -1 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & 0 & -1 & 1 & 0
\end{matrix}$$

$E1 \leftarrow E1 + E6$

$E2 \leftarrow E2 - E6$

$\rightsquigarrow$

$E4 \leftarrow E4 + E6$

$E5 \leftarrow E5 - E6$

p	q	m	s	t	u	=	A	B	C	D	E	F
1	0	0	0	0	0		0	1	-1	0	1	-1
0	1	0	0	0	0		1	-1	1	0	-1	1
0	0	1	0	0	0		-1	1	0	0	0	0
0	0	0	1	0	0		0	0	0	0	1	-1
0	0	0	0	1	0		1	-1	0	1	-1	1
0	0	0	0	0	1		-1	1	0	-1	1	0

$p = B - C + E - F$

$q = A - B + C - E + F$

$m = -A + B$

$s = E - F$

$t = A - B + D - E + F$.

$u = -A + B - D + E$

Controllo: $p + q =$ $B - C + E - F$

$+ A - B + C - E + F$

$= A$

e così via.