DA FOTOCOPIARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 27 settembre 2019

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente. I compiti corretti saranno a disposizione martedì 1 ottobre, alle 11.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r\operatorname{sen}\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Le formule per le coordinate sferiche sono:

$$x = r\operatorname{sen}\theta\cos\phi \qquad y = r\operatorname{sen}\theta\operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\operatorname{sen}\theta\,dr\,d\theta\,d\phi$$

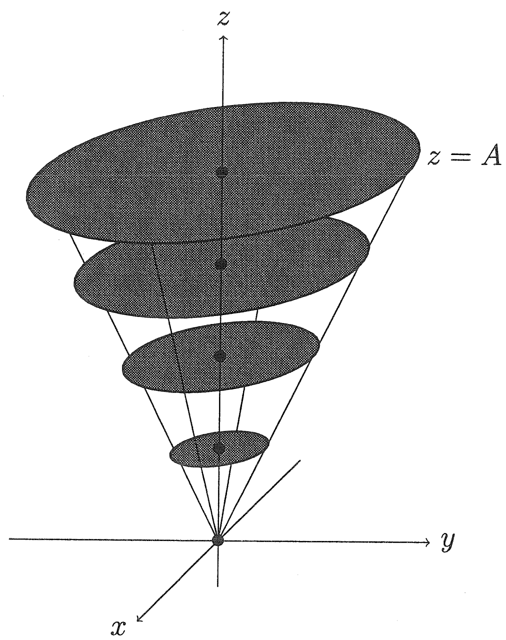
A. Usare la sostituzione $x = \operatorname{sen} t$ e calcolare

$$\int \frac{1}{\cos^3 t}\,dt$$

B. Siano $A, R > 0$ due parametri costanti. Sia $\mathcal{C}$ il cono con vertice all'origine, con asse centrale nella direzione dell'asse z positivo, e con soffitto che è un disco di raggio R che si trova al livello $z = A$. Usare le coordinate cilindriche e calcolare la media:

$$\overline{x^2 y^2 z^2}$$

sul cono $\mathcal{C}$.



C. Usare le coordinate *sferiche* e calcolare il volume della forma definita da:

$$0 \leq z \leq y \leq x \leq 1$$

D. Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio delle soluzioni del sistema:

$$\begin{array}{rrrrl} A & +3B & +6C & +10D & = 0 \\ A & +B & +C & +D & = 1 \\ A & +2B & +3C & +4D & = 2 \\ 3A & +6B & +10C & +15D & = 3 \end{array}$$

(Indicazione: nella risposta finale non ci sono denominatori.)

$\boxed{A}$ $x = \text{sen}\, t$

$dx = \cos t \, dt \rightarrow dt = dx / \cos t$

$$\int \frac{1}{\cos^3 t} dt = \int \frac{1}{\cos^3 t} \frac{dx}{\cos t}$$

$$= \int \frac{1}{\cos^4 t} dx = \int \frac{1}{(\cos^2 t)^2} dx$$

$$= \int \frac{1}{(1 - \text{sen}^2(t))^2} dx = \int \frac{1}{(1 - x^2)^2} dx$$

$$\frac{1}{(1-x^2)^2} = \frac{1}{(1-x)^2 (1+x)^2}$$

$$= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1+x)} + \frac{D}{(1+x)^2}$$

$$\rightsquigarrow 1 = A(1-x)(1+x)^2 + B(1+x)^2$$
$$+ C(1-x)^2(1+x) + D(1-x)^2$$

$$x = 1 \rightsquigarrow 1 = 0 + B(1+1)^2 + 0 + 0 = 4B$$
$$\to B = 1/4$$
$$x = -1 \rightsquigarrow 1 = 0 + 0 + 0 + D(1-(-1))^2$$
$$= 4D \rightsquigarrow D = 1/4$$

$$1 = A(1-x)(1+x)^2 + B(1+x)^2$$
$$+ C(1-x)^2(1+x) + D(1-x)^2$$

$$= A(1-x^2)(1+x) + \frac{1}{4}(1+2x+x^2)$$
$$+ C(1-x^2)(1-x) + \frac{1}{4}(1-2x+x^2)$$

$$= A(1+x-x^2-x^3) + \frac{1}{4}(1+2x+x^2)$$
$$+ C(1-x-x^2+x^3) + \frac{1}{4}(1-2x+x^2)$$

$$= (A + \frac{1}{4} + C + \frac{1}{4})$$
$$+ (A + \frac{1}{2} - C - \frac{1}{2})x$$
$$+ (-A + \frac{1}{4} - C + \frac{1}{4})x^2$$
$$+ (-A + C)x^3$$

$$\rightarrow \quad A + C + \frac{1}{2} = 1 \quad \rightsquigarrow \quad A + C = \frac{1}{2}$$
$$A - C = 0 \qquad A = C$$
$$-A - C + \frac{1}{2} = 0$$
$$-A + C = 0$$

$$\rightsquigarrow \quad A = C = \frac{1}{4}$$

$$\int \frac{1}{4}\frac{1}{1-x} + \frac{1}{4}\frac{1}{(1-x)^2} + \frac{1}{4}\frac{1}{1+x} + \frac{1}{4}\frac{1}{(1+x)^2}\,dx$$

$$= \frac{1}{4}\left(-\log(1-x) + \frac{1}{1-x} + \log(1+x) - \frac{1}{1+x}\right)$$

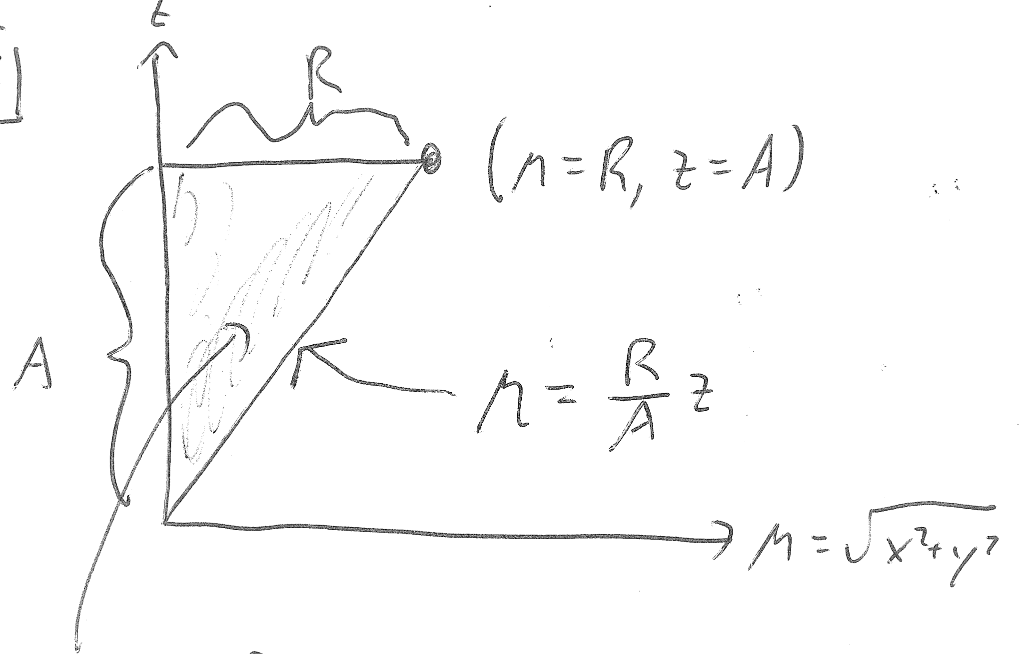
$$= \frac{1}{4}\left(-\log(1-\operatorname{sen} t) + \frac{1}{1-\operatorname{sen} t} + \log(1+\operatorname{sen} t) - \frac{1}{1+\operatorname{sen} t}\right)$$

$$= \frac{1}{4}\left(\log\left(\frac{1+\operatorname{sen} t}{1-\operatorname{sen} t}\right) + \frac{(1+\operatorname{sen} t) - (1-\operatorname{sen} t)}{(1-\operatorname{sen} t)(1+\operatorname{sen} t)}\right)$$

$$= \frac{1}{4}\log\left(\frac{1+\operatorname{sen} t}{1-\operatorname{sen} t}\right) + \frac{1}{2}\frac{\operatorname{sen} t}{1-\operatorname{sen}^2 t}$$

$$= \frac{1}{4}\log\left(\frac{1+\operatorname{sen} t}{1-\operatorname{sen} t}\right) + \frac{1}{2}\frac{\operatorname{sen} t}{\cos^2 t}$$

$\boxed{B}$



$$r \le \frac{R}{A} z$$

$$\int_{\theta=0}^{2\pi} \left(\int_{z=0}^{A} \left(\int_{r=0}^{R/A \cdot z} r \, dr \right) dz \right) d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{z=0}^{A} \frac{1}{2} \left(\frac{R}{A} z \right)^2 dz \right) d\theta$$

$$= \frac{1}{2} \frac{R^2}{A^2} \int_{\theta=0}^{2\pi} \frac{1}{3} A^3 \, d\theta = 2\pi \cdot \frac{1}{6} \frac{R^2 A^3}{A^2}$$

$$= \frac{\pi}{3} R^2 A,$$ il volume del cono

$$\int_0^{2\pi}\left(\int_0^{A}\left(\int_0^{R/A\cdot z}(r\cos\theta)^2(r\operatorname{sen}\theta)^2 z^2\cdot r\,dr\right)dz\right)d\theta$$

$$=\int_0^{2\pi}\cos^2\theta\operatorname{sen}^2\theta\left(\int_0^{A} z^2\left(\int_0^{R/A\cdot z} r^5\,dr\right)dz\right)d\theta$$

$$=\int_0^{2\pi}\cos^2\theta\operatorname{sen}^2\theta\left(\int_0^{A} z^2\cdot\frac{1}{6}\left(\frac{R}{A}\cdot z\right)^6 dz\right)d\theta$$

$$=\frac{1}{6}\frac{R^6}{A^6}\int_0^{2\pi}\cos^2\theta\operatorname{sen}^2\theta\left(\int_0^{A} z^8\,dz\right)d\theta$$

$$=\frac{1}{6}\frac{R^6}{A^6}\int_0^{2\pi}\cos^2\theta\operatorname{sen}^2\theta\;\frac{1}{9}A^9\,d\theta$$

$$=\frac{1}{54}R^6A^3\int_0^{2\pi}\cos^2\theta\operatorname{sen}^2\theta\,d\theta$$

p. 6/12

$$\left[\int_0^{2\pi} \cos^2\theta \, \mathrm{sen}^2\theta \, d\theta \right.$$

$$= \int_0^{2\pi} \cos^2\theta \, (1 - \cos^2\theta) \, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta - \cos^4\theta \, d\theta$$

$$\int_0^{2\pi} \cos^2\theta = \frac{1}{2} \cdot 2\pi = \pi$$

$$\int_0^{2\pi} \cos^4\theta = \frac{1}{4} \, \mathrm{sen}\,\theta \cos^3\theta \Big]_{\theta=0}^{2\pi} + \frac{3}{4} \int_0^{2\pi} \cos^2\theta \, d\theta$$

$$= 0 + \frac{3}{4}\pi = \frac{3}{4}\pi$$

$$\left. \int_0^{2\pi} \cos^2\theta \, \mathrm{sen}^2\theta \, d\theta = \pi - \frac{3}{4}\pi = \frac{1}{4}\pi \right]$$

$$\iiint_{\text{Cono}} x^2 y^2 z^2 \, dV = \frac{1}{54} R^6 A^3 \cdot \frac{1}{4}\pi = \frac{1}{216} \pi R^6 A^3$$

$$\overline{x^2 y^2 z^2} = \frac{\frac{1}{216} \pi R^6 A^3}{\frac{1}{3} \pi R^2 A} = \frac{1}{72} R^4 A^2$$

$\boxed{C}$ $0 \leq z \to r\cos\theta \geq 0 \to \cos\theta \geq 0$

$\to 0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq x, y \to 0 \leq r \operatorname{sen}\theta \cos\varphi$

$0 \leq r \operatorname{sen}\theta \operatorname{sen}\varphi$

$\to \operatorname{sen}\varphi, \cos\varphi \geq 0 \to 0 \leq \varphi \leq \frac{\pi}{2}$

$x \leq 1 \to r \operatorname{sen}\theta \cos\varphi \leq 1$

$\to r \leq 1/\operatorname{sen}\theta\cos\varphi$

$y \leq x \to r \operatorname{sen}\theta \operatorname{sen}\varphi \leq r \operatorname{sen}\theta \cos\varphi$

$\to \operatorname{sen}\varphi \leq \cos\varphi$

$\to \operatorname{tg}\varphi \leq 1 \to \varphi \leq \operatorname{arctg}(1) = \frac{\pi}{4}$

$\to 0 \leq \varphi \leq \pi/4$

$z \leq y \to r\cos\theta \leq r \operatorname{sen}\theta \operatorname{sen}\varphi$

$\to \cos\theta \leq \operatorname{sen}\theta \operatorname{sen}\varphi$

$\to 1/\operatorname{sen}\varphi \leq \operatorname{tg}\theta$

$\to \operatorname{arctg}(1/\operatorname{sen}\varphi) \leq \theta$

$\to \operatorname{arctg}(1/\operatorname{sen}\varphi) \leq \theta \leq \frac{\pi}{2}$

$$\iiint\limits_{0\le z\le y\le x\le 1} 1\, dV$$

$$= \int_{\varphi=0}^{\pi/4} \left(\int_{\theta=\operatorname{arcctg}(1/\operatorname{sen}\varphi)}^{\pi/2} \left(\int_{r=0}^{1/\operatorname{sen}\theta\cos\varphi} r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/4} \left(\int_{\theta=\operatorname{arcctg}(1/\operatorname{sen}\varphi)}^{\pi/2} \frac{1}{3} \left(\frac{1}{\operatorname{sen}\theta\cos\varphi} \right)^3 \operatorname{sen}\theta \, d\theta \right) d\varphi$$

$$= \frac{1}{3} \int_{\varphi=0}^{\pi/4} \frac{1}{\cos^3\varphi} \left(\int_{\theta=\operatorname{arcctg}(1/\operatorname{sen}\varphi)}^{\pi/2} \csc^2\theta \, d\theta \right) d\varphi$$

$$= \frac{1}{3} \int_{\varphi=0}^{\pi/4} \frac{1}{\cos^3\varphi} \Big[-\operatorname{ctg}\theta \Big]_{\theta=\operatorname{arcctg}(1/\operatorname{sen}\varphi)}^{0} d\varphi$$

$$= \frac{1}{3} \int_{\varphi=0}^{\pi/4} \frac{1}{\cos^3\varphi} \left(0 - \left(-\operatorname{ctg}(\operatorname{arcctg}(1/\operatorname{sen}\varphi)\right)\right) d\varphi$$

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$$= \frac{1}{3} \int_{\varphi=0}^{\pi/4} \frac{1}{\cos^3\varphi} \operatorname{sen}\varphi \, d\varphi$$

(poiché tg e ctg sono inversi)

$$= \frac{1}{3} \frac{1}{2} (\cos\varphi)^{-2} \Big]_{\varphi=0}^{\pi/4}$$

$$= \frac{1}{6} \left(\frac{1}{(\sqrt{2}/2)^2} - \frac{1}{1^2} \right) = \frac{1}{6}(2-1) = \frac{1}{6}$$

$\boxed{D}$

$$\begin{array}{cccccc} A & B & C & D & = & \\ 1 & 3 & 6 & 10 & & 0 \\ 1 & 1 & 1 & 1 & & 1 \\ 1 & 2 & 3 & 4 & & 2 \\ 3 & 6 & 10 & 15 & & 3 \end{array}$$

$$\xrightarrow[\substack{E3 \leftarrow E3 - E1 \\ E4 \leftarrow E4 - 3E1}]{E2 \leftarrow E2 - E1} \begin{array}{ccccc} 1 & 3 & 6 & 10 & 0 \\ 0 & -2 & -5 & -9 & 1 \\ 0 & -1 & -3 & -6 & 2 \\ 0 & -3 & -8 & -15 & 3 \end{array}$$

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$$\xrightarrow{E2 \leftarrow -\frac{1}{2}E2} \begin{pmatrix} 1 & 3 & 6 & 10 & 0 \\ 0 & 1 & 5/2 & 9/2 & -1/2 \\ 0 & -1 & -3 & -6 & 2 \\ 0 & -3 & -8 & -15 & 3 \end{pmatrix}$$

$$\xrightarrow{E1 \leftarrow E1 - 3E2} \begin{pmatrix} 1 & 0 & -3/2 & -7/2 & 3/2 \\ 0 & 1 & 5/2 & 9/2 & -1/2 \\ 0 & 0 & -1/2 & -3/2 & 3/2 \\ 0 & 0 & -1/2 & -3/2 & 3/2 \end{pmatrix}$$

$E3 \leftarrow E3 + E2$

$E4 \leftarrow E4 + 3E2$

$$\xrightarrow{E3 \leftarrow -2E3} \begin{pmatrix} 1 & 0 & -3/2 & -7/2 & 3/2 \\ 0 & 1 & 5/2 & 9/2 & -1/2 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & -1/2 & -3/2 & 3/2 \end{pmatrix}$$

$$\xrightarrow{E1 \leftarrow E1 + 3/2E3} \begin{pmatrix} 1 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$E2 \leftarrow E2 - 5/2E3$

$E4 \leftarrow E4 + \frac{1}{2}E3$

p. 11/12

$$
\begin{aligned}
A \quad\quad\quad\quad & + D = -3 \\
B \quad\quad & -3D = 7 \\
C & +3D = -3 \\
& 0 = 0
\end{aligned}
$$

$$
\rightarrow \boxed{\begin{aligned}
A &= -D - 3 \\
B &= 3D + 7 \\
C &= -3D - 3 \\
D &= D
\end{aligned}} = D \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 7 \\ -3 \\ 0 \end{pmatrix}
$$

<u>Controllo:</u>

$$
\begin{aligned}
A + 3B + 6C + 10D = & \,(-D-3) \\
& + 3(3D+7) \\
& + 6(-3D-3) \\
& + 10D
\end{aligned}
$$

$$
= D(-1+9-18+10) + (-3+21-18) = 0
$$

$$
\begin{aligned}
A+B+C+D &= (-D-3) \\
&\quad + (3D+7) \\
&\quad + (-3D-3) \\
&\quad + D \\
&= D(-1+3-3+1) + (-3+7-3) = 1
\end{aligned}
$$

$$
\begin{aligned}
A+2B+3C+4D &= (-D-3) \\
&\quad + 2(3D+7) \\
&\quad + 3(-3D-3) \\
&\quad + 4D \\
&= D(-1+6-9+4) + (-3+14-9) = 2
\end{aligned}
$$

$$
\begin{aligned}
3A+6B+10C+15D &= 3(-D-3) \\
&\quad + 6(3D+7) \\
&\quad + 10(-3D-3) \\
&\quad + 15D \\
&= D(-3+18-30+15) + (-9+42-30) = 3
\end{aligned}
$$