DA FOTOCOPIARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 12 luglio 2019

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente. I compiti corretti saranno a disposizione martedì 16 luglio, alle 9.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Le formule per le coordinate sferiche sono:

$$x = r\,\mathrm{sen}\,\theta\cos\phi \qquad y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\,\mathrm{sen}\,\theta\,dr\,d\theta\,d\phi$$

A. Calcolare

$$\int \frac{Q^2}{(Q^2-2Q+2)(Q^2+2Q+2)}\,dQ$$

B. Sia $R > 0$ un parametro fisso. Sia C il cilindro infinito definito da

$$x^2 + y^2 \leq R^2$$

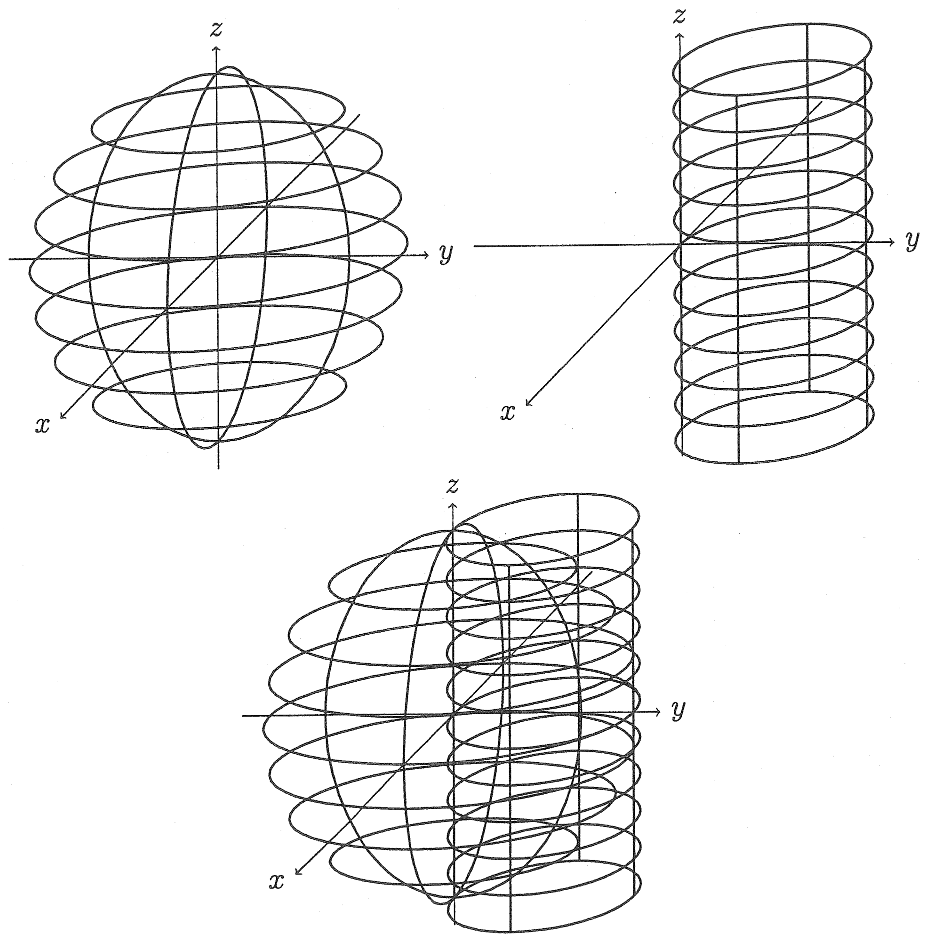
Usare le coordinate *sferiche* e calcolare

$$\iiint_C \frac{1}{x^2+y^2+z^2}\,dV$$

C. Usare le coordinate *cilindriche* e calcolare il volume della forma definita da:

$$2x^2 + y^2 + z^2 \leq 4 \qquad\qquad x^2 + (y-1)^2 \leq 1$$

Indicazioni: il volume d'interesse è l'intersezione. Conviene che la prima integrale da svolgere sia l'integrale dz.



D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema

$$\begin{array}{rrrl} x & +y & +z & = p \\ 2x & +3y & +4z & = q \\ 3x & +6y & +10z & = r \end{array}$$

per le incognite x, y, e z in termini di p, q, e r.

$\boxed{A}$

$$\frac{Q^2}{(Q^2-2Q+2)(Q^2+2Q+2)}$$

$$= \frac{A}{Q^2-2Q+2} + \frac{B(2Q-2)}{Q^2+2Q+2}$$

$$+ \frac{C}{Q^2+2Q+2} + \frac{D(2Q+2)}{Q^2+2Q+2}$$

$$\hookrightarrow Q^2 = A(Q^2+2Q+2)$$
$$+ B(2Q-2)(Q^2+2Q+2)$$
$$+ C(Q^2-2Q+2)$$
$$+ D(2Q+2)(Q^2-2Q+2)$$

$$= A(\quad Q^2+2Q+2)$$
$$+ B(2Q^3+2Q^2 \quad -4)$$
$$+ C(\quad Q^2-2Q+2)$$
$$+ D(2Q^3-2Q^2 \quad +4)$$

$$\rightsquigarrow \quad 2B+2D=0$$
$$A+2B+C-2D=1$$
$$2A-2C=0$$
$$2A-4B+2C+4D=0$$

Dalla 1ª e la 3ª equazioni si trova $D=-B$, $C=A$.
La 2ª e la 4ª equazioni diventano

$$A+2B+A+2B=1 \quad \rightsquigarrow \quad 2A+4B=1$$
$$2A-4B+2A-4B=0 \qquad\quad 2A-4B=0$$

Se si fanno la somma e la differenza di queste ultime due si trova

$$4A=1 \quad \rightsquigarrow \quad A=1/4 \quad \longrightarrow \quad C=1/4$$
$$8B=1 \qquad\quad B=1/8 \qquad\quad D=-1/8$$

$$\int \frac{1}{4}\frac{1}{Q^2-2Q+2} + \frac{1}{8}\frac{2Q-2}{Q^2-2Q+2}$$

$$+\frac{1}{4}\frac{1}{Q^2+2Q+2} - \frac{1}{8}\frac{2Q+2}{Q^2+2Q+2}\, dQ$$

$$= \frac{1}{4}\frac{2}{\sqrt{4}} \arctan\left(\frac{2Q-2}{\sqrt{4}}\right)$$

$$+ \frac{1}{8}\log(Q^2-2Q+2)$$

$$+ \frac{1}{4}\frac{2}{\sqrt{4}} \arctan\left(\frac{2Q+2}{\sqrt{4}}\right)$$

$$- \frac{1}{8}\log(Q^2+2Q+2)$$

$$= \frac{1}{4}\arctan(Q-1) + \frac{1}{4}\arctan(Q+1)$$

$$+ \frac{1}{8}\log(Q^2-2Q+2) - \frac{1}{8}\log(Q^2+2Q+2)$$

$\boxed{B}$ $x^2 + y^2 \le R^2$

$\hookrightarrow r^2 \operatorname{sen}^2\theta \cos^2\varphi + r^2 \operatorname{sen}^2\theta \operatorname{sen}^2\varphi \le R^2$

$\hookrightarrow r^2 \operatorname{sen}^2\theta \le R^2$

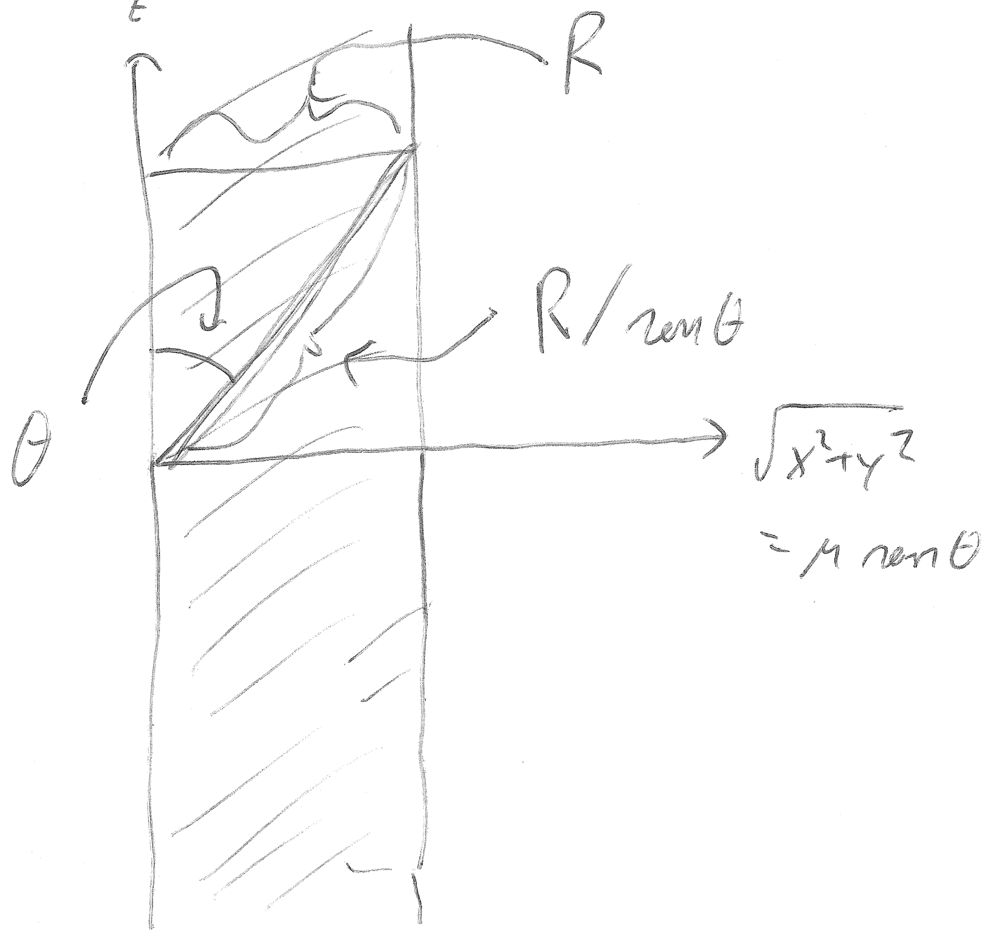
$\hookrightarrow r \operatorname{sen}\theta \le R \rightsquigarrow r \le R/\operatorname{sen}\theta$

$$\iiint_{C} \frac{1}{x^2+y^2+z^2}\, dV$$

$$= \int_{\varphi=0}^{2\pi} \left(\int_{\theta=0}^{\pi} \left(\int_{r=0}^{R/\operatorname{sen}\theta} \frac{1}{r^2} \cdot r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \left(\int_{\theta=0}^{\pi} \frac{R}{\operatorname{sen}\theta} \cdot \operatorname{sen}\theta \, d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \pi R \, d\varphi = 2\pi^2 R$$

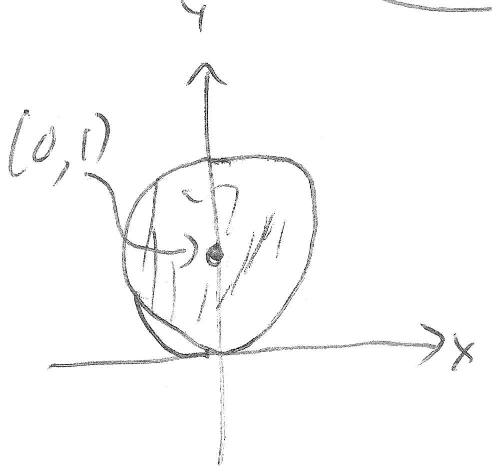


$\boxed{C}$ $x^2+(y-1)^2 \le 1$

$\hookrightarrow x^2+y^2-2y+1 \le 1$

$\hookrightarrow x^2+y^2 \le 2y \rightsquigarrow r^2 \le 2r\operatorname{sen}\theta$

$\hookrightarrow r \le 2\operatorname{sen}\theta$



$\rightsquigarrow 0 \le \theta \le \pi$

$$2x^2+y^2+z^2 \le 4 \rightsquigarrow z^2 \le 4-2x^2-y^2$$

$$\rightsquigarrow |z| \le \sqrt{4-2x^2-y^2}$$

$$= \sqrt{4-2r^2\cos^2\theta - r^2 \operatorname{sen}^2\theta}$$

$$\iiint\limits_{\substack{2x^2+y^2+z^2\le 4 \\ x^2+(y-1)^2\le 1}} 1 \, d\bar{V}$$

$$= \int_{\theta=0}^{\pi} \left(\int_{r=0}^{2\operatorname{sen}\theta} \left(\int_{z=-\sqrt{4-2r^2\cos^2\theta-r^2\operatorname{sen}^2\theta}}^{+\sqrt{4-2r^2\cos^2\theta-r^2\operatorname{sen}^2\theta}} r \, dz \right) dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\int_{r=0}^{2 \operatorname{sen} \theta} 2r \sqrt{4 - (2\cos^2\theta + \operatorname{sen}^2\theta) r^2} \, dr \right) d\theta$$

$$= \frac{2}{3} \int_{\theta=0}^{\pi} \frac{-1}{2\cos^2\theta + \operatorname{sen}^2\theta} \left(4 - (2\cos^2\theta + \operatorname{sen}^2\theta) r^2\right)^{3/2} \Big]_{r=0}^{2\operatorname{sen}\theta} d\theta$$

$$= \frac{2}{3} \int_{\theta=0}^{\pi} \frac{-1}{2\cos^2\theta + \operatorname{sen}^2\theta} \left(4 - (2\cos^2\theta + \operatorname{sen}^2\theta) 4 \operatorname{sen}^2\theta\right)^{3/2}$$

$$+ \frac{1}{2\cos^2\theta + \operatorname{sen}^2\theta} (4 - 0)^{3/2} \, d\theta$$

$$= \frac{2}{3} \int_{\theta=0}^{\pi} \frac{-1}{2 - \operatorname{sen}^2\theta} \left(4 - 4(2 - \operatorname{sen}^2\theta) \operatorname{sen}^2\theta\right)^{3/2}$$

$$+ \frac{1}{2 - \operatorname{sen}^2\theta} 4^{3/2} \, d\theta$$

p. 8/11

$$= \frac{2}{3} 4^{3/2} \int_{\theta=0}^{\pi} \frac{-1}{2-\operatorname{sen}^2\theta} \left(1 - 2\operatorname{sen}^2\theta + \operatorname{sen}^4\theta\right)^{3/2} + \frac{1}{2-\operatorname{sen}^2\theta} \, d\theta$$

$$= \frac{2}{3} 8 \int_{\theta=0}^{\pi} -\frac{1}{2-\operatorname{sen}^2\theta} \left(\left(1 - \operatorname{sen}^2\theta\right)^2\right)^{3/2} \frac{1}{2-\operatorname{sen}^2\theta} \, d\theta$$

$$= \frac{2}{3} 8 \int_{\theta=0}^{\pi} - \frac{|1-\operatorname{sen}^2\theta|^3}{2-\operatorname{sen}^2\theta} + \frac{1}{2-\operatorname{sen}^2\theta} \, d\theta$$

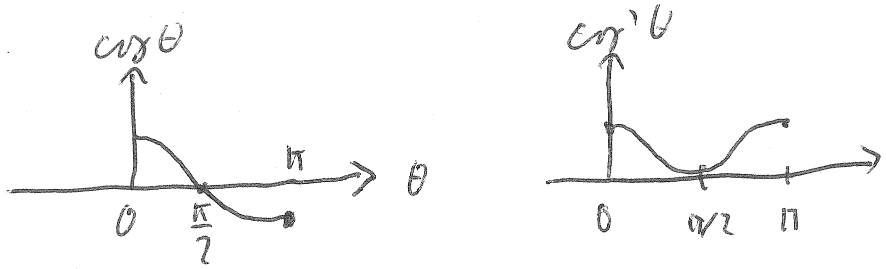
$$= \frac{2}{3} 8 \int_{\theta=0}^{\pi} - \frac{\cos^6\theta}{2-\operatorname{sen}^2\theta} + \frac{1}{2-\operatorname{sen}^2\theta} \, d\theta$$

$$= \frac{2}{3} 8 \int_{\theta=0}^{\pi} - \frac{\cos^6\theta}{1+\cos^2\theta} + \frac{1}{1+\cos^2\theta} \, d\theta$$

p. 8 bis/11

Il resto del calcolo è piuttosto lungo e sopratutto piuttosto difficile.

$$\int_{\theta=0}^{\pi} \frac{1}{1+\cos^2\theta}\,d\theta = 2\int_{\theta=0}^{\pi/2} \frac{1}{1+\cos^2\theta}\,d\theta$$



$t = tg\,\theta \qquad dt = \sec^2\theta\, d\theta$

$$= 2\int_{\theta=0}^{\pi/2} \frac{1}{(1+\cos^2\theta)}\frac{1}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} \frac{1}{\sec^2\theta+1}\sec^2\theta\, d\theta$$

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$$= 2\int_{t=0}^{+\infty} \frac{1}{(t^2+1)+1}\, dt$$

$$= 2\int_{t=0}^{+\infty} \frac{1}{t^2+2}\, dt$$

$$= 2\,\frac{1}{\sqrt{2}} \arctan\left(\frac{t}{\sqrt{2}}\right)\Bigg]_{t=0}^{+\infty}$$

$$= \frac{2}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{2}{\sqrt{2}}\,\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}$$

$$\int_{\theta=0}^{\pi} \frac{\cos^6\theta + 1}{\cos^2\theta + 1}\, d\theta$$

$$= \int_{\theta=0}^{\pi} \cos^4\theta - \cos^2\theta + 1\, d\theta$$

$$= \frac{3}{4}\cdot\frac{1}{2}\pi - \frac{1}{2}\pi + \pi$$

$$= \left(\frac{3}{8} - \frac{4}{8} + \frac{8}{8}\right)\pi = \frac{7}{8}\pi$$

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$$\int_{\theta=0}^{\pi} \frac{\cos^6\theta}{\cos^2\theta+1}\,d\theta = \int_{\theta=0}^{\pi} \frac{\cos^6\theta+1}{\cos^2\theta+1}\,d\theta - \int_{\theta=0}^{\pi} \frac{1}{\cos^2\theta+1}\,d\theta$$

$$= \frac{7}{8}\pi - \frac{1}{\sqrt{2}}\pi = \left(\frac{7}{8} - \frac{1}{\sqrt{2}}\right)\pi$$

$$\iiint\limits_{\substack{2x^2+y^2+z^2\le 4 \\ x^2+(y-1)^2\le 1}} 1\,dV = \frac{2}{3}\cdot 8\left(-\left(\frac{7}{8}-\frac{1}{\sqrt{2}}\right)\pi + \frac{1}{\sqrt{2}}\pi\right)$$

$$= \frac{16\pi}{3}\left(-\frac{7}{8} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{16\pi}{3}\left(\sqrt{2} - \frac{7}{8}\right)$$

$$= \frac{2\pi}{3}\left(8\sqrt{2} - 7\right)$$

$\boxed{b}$

x	y	z	$=$	p	q	r
1	1	1		1	0	0
2	3	4		0	1	0
3	6	10		0	0	1

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$$\xrightarrow[E3 \leftarrow E3 - 3E1]{E2 \leftarrow E2 - 2E1} \begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 7 & -3 & 0 & 1 \end{array}$$

$$\xrightarrow[E3 \leftarrow E3 - 2E2]{E1 \leftarrow E1 - E2} \begin{array}{cccccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}$$

$$\xrightarrow[E2 \leftarrow E2 - 2E3]{E1 \leftarrow E1 + E3} \begin{array}{ccc|ccc} x & y & z = & p & q & r \\ \hline 1 & 0 & 0 & 6 & -4 & 1 \\ 0 & 1 & 0 & -8 & 7 & -2 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}$$

$$\begin{aligned} x &= 6p - 4q + r \\ y &= -8p + 7q - 2r \\ z &= 3p - 3q + r \end{aligned}$$

Controllo =

$$x+y+z = \begin{aligned}&6p-4q+r\\&-8p+7q-2r\\&+3p-3q+r\end{aligned} = p$$

$$2x+3y+4z = \begin{aligned}&2(6p-4q+r)\\&+3(-8p+7q-2r)\\&+4(3p-3q+r)\end{aligned} = \begin{aligned}&12p-8q+2r\\&-24p+21q-6r\\&+12p-12q+4r\end{aligned}$$

$$= 0p+1q+0r$$

$$= q$$

$3x+6x+10z$

$$= \begin{aligned}&3(6p-4q+r)\\&+6(-8p+7q-2r)\\&+10(3p-3q+r)\end{aligned} = \begin{aligned}&18p-12q+3r\\&-48p+42q-12r\\&+30p-30q+10r\end{aligned}$$

$$= 0p+0q+1r = r$$