DA FOTOCOPIARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 19 giugno 2019

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente. I compiti corretti saranno a disposizione martedì 25 giugno, alle 9.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Le formule per le coordinate sferiche sono:

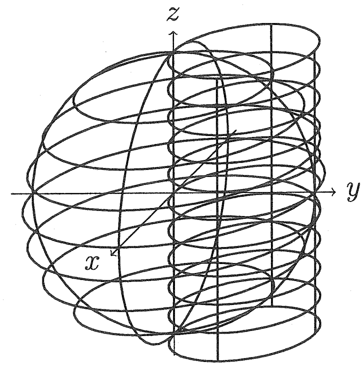
$$x = r\operatorname{sen}\theta\cos\phi \qquad y = r\operatorname{sen}\theta\operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\operatorname{sen}\theta\,dr\,d\theta\,d\phi$$

A. Usare la sostituzione $z = y^2$ e calcolare

$$\int \frac{\sqrt{z}}{z^2+5z+4}\,dz$$

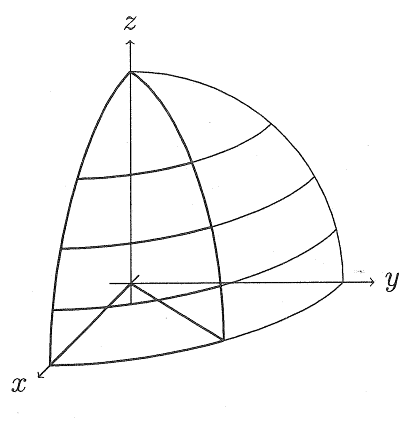
B. Usare le coordinate *cilindriche* e calcolare il volume della forma definita da:

$$x^2 + y^2 + z^2 \leq 4 \qquad\qquad x^2 + (y-1)^2 \leq 1$$



C. Sia $R > 0$ un parametro fisso. Usare le coordinate *sferiche* e calcolare la media di x sulla forma definita da:

$$x, y, z \geq 0 \qquad\qquad x \geq y \qquad\qquad x^2 + y^2 + z^2 \leq R^2$$



D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema

$$\begin{array}{cccc} x & +y & +z & = A \\ x & +3y & +5z & = B \\ x & +6y & +15z & = C \end{array}$$

per le incognite x, y, e z in termini di A, B, e C.

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$\boxed{A}$

$z = y^2 \qquad y = \sqrt{z}$

$dz = 2y\,dy$

$$\int \frac{\sqrt{z}}{z^2+5z+4}\,dz = \int \frac{y}{y^4+5y^2+4}\,2y\,dy$$

$$= \int \frac{2y^2}{(y^2+1)(y^2+4)}\,dy$$

$$\frac{2y^2}{(y^2+1)(y^2+4)} = \frac{A}{y^2+1} + \frac{B(2y)}{y^2+1} + \frac{C}{y^2+4} + \frac{D(2y)}{y^2+4}$$

$$2y^2 = A(y^2+4) + B\cdot 2y\cdot(y^2+4) + C(y^2+1) + D\cdot 2y\cdot(y^2+1)$$

$$\begin{aligned} &= (2B+2D)y^3 \\ &\quad + (A+C)y^2 \\ &\quad + (8B+2D)y \\ &\quad + (4A+C) \end{aligned}$$

$$\left.\begin{array}{l} 2B + D = 0 \\ 8B + 2D = 0 \end{array}\right\} \rightsquigarrow B = D = 0$$

$$\begin{array}{l} A + C = 2 \\ 4A + C = 0 \end{array} \rightsquigarrow \begin{array}{l} A + C = 2 \\ -3C = -8 \end{array}$$

$$\rightsquigarrow \begin{array}{l} A + C = 2 \\ C = 8/3 \end{array} \rightsquigarrow \begin{array}{l} A = -2/3 \\ C = 8/3 \end{array}$$

$$\int \frac{2y^2}{(y^2+1)(y^2+4)} dy = \int -\frac{2}{3} \frac{1}{y^2+1} + \frac{8}{3} \frac{1}{y^2+4} dy$$

$$= -\frac{2}{3} \operatorname{arctg}(y) + \frac{8}{3} \cdot \frac{1}{2} \operatorname{arctg}\left(\frac{y}{2}\right)$$

$$= -\frac{2}{3} \operatorname{arctg}(\sqrt{z}) + \frac{4}{3} \operatorname{arctg}\left(\frac{\sqrt{z}}{2}\right)$$

$\boxed{B}$ $x^2 + y^2 + z^2 \leq 4 \rightsquigarrow r^2 + z^2 \leq 4$

$$\rightsquigarrow z^2 \leq 4 - r^2 \rightsquigarrow -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$$

$$x^2 + (y-1)^2 \leq 1 \rightsquigarrow x^2 + y^2 - 2y + 1 \leq 1$$

$$\rightsquigarrow x^2 + y^2 \leq 2y \rightsquigarrow r^2 \leq 2r \operatorname{sen}\theta$$

$$\rightsquigarrow r \leq 2 \operatorname{sen}\theta$$

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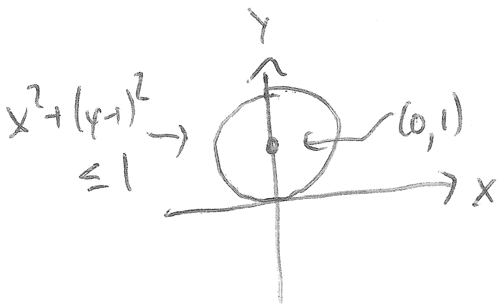

$0 \le \theta \le \pi$

$$\iiint\limits_{\substack{x^2+y^2+z^2 \le 4 \\ x^2+(y-1)^2 \le 1}} 1 \, dV$$

$$= \int_{\theta=0}^{\pi} \left(\int_{r=0}^{2\,\mathrm{sen}\,\theta} \left(\int_{z=-\sqrt{4-r^2}}^{+\sqrt{4-r^2}} 1 \cdot r \, dz \right) dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\int_{r=0}^{2\,\mathrm{sen}\,\theta} 2r\sqrt{4-r^2} \, dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi} -\frac{2}{3}\left(4-r^2\right)^{3/2} \Big]_{r=0}^{2\,\mathrm{sen}\,\theta} d\theta$$

$$= \int_{\theta=0}^{\pi} -\frac{2}{3}\left(4-4\,\mathrm{sen}^2\theta\right)^{3/2} + \frac{2}{3}\, 4^{3/2} \, d\theta$$

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$$= \int_{\theta=0}^{\pi} -\frac{2}{3}\left(4\cos^2\theta\right)^{3/2} + \frac{2}{3}\, 8 \, d\theta$$

$$= \frac{16}{3}\int_{\theta=0}^{\pi} -|\cos\theta|^3 + 1 \, d\theta$$

$$\Big[\quad \int \cos^3\theta \, d\theta = \frac{1}{3}\operatorname{sen}\theta\cos^2\theta + \frac{2}{3}\operatorname{sen}\theta$$

$$\int_{\theta=0}^{\pi/2} \cos^3\theta \, d\theta = \frac{2}{3}$$

$$\int_{\theta=\pi/2}^{\pi} \cos^3\theta = -\frac{2}{3} \rightarrow \int_{\pi/2}^{\pi} |\cos^3\theta| \, d\theta = \frac{2}{3}$$

$$\int_0^{\pi} |\cos^3\theta| \, d\theta = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \quad \Big]$$

$$= \frac{16}{3}\left(-\frac{4}{3} + \pi\right)$$

$$= \frac{16}{9}(3\pi - 4)$$

$\boxed{C}$

$x^2+y^2+z^2 \le R^2 \rightsquigarrow 0 \le r \le R$

$z \ge 0 \to r \cos\theta \ge 0 \rightsquigarrow \cos\theta \ge 0$

$\to 0 \le \theta \le \frac{\pi}{2}$

$x, y \ge 0 \to r \operatorname{sen}\theta \cos\varphi \ge 0$
$r \operatorname{sen}\theta \operatorname{sen}\varphi \ge 0$

$\to \operatorname{sen}\varphi, \cos\varphi \ge 0 \to 0 \le \varphi \le \frac{\pi}{2}$

$x \ge y \to r \operatorname{sen}\theta \cos\varphi \ge r \operatorname{sen}\theta \operatorname{sen}\varphi$

$\rightsquigarrow \cos\varphi \ge \operatorname{sen}\varphi$

$\rightsquigarrow 1 \ge \operatorname{tg}\varphi$

$\rightsquigarrow \frac{\pi}{4} \ge \varphi$

$$\iiint\limits_{\substack{x^2+y^2+z^2\le R^2 \\ x,y,z \ge 0 \\ x \ge y}} 1 \, dV = \int_{\varphi=0}^{\pi/4} \left(\int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{R} r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/4} \left(\int_{\theta=0}^{\pi/2} \frac{R^3}{3} \operatorname{sen}\theta \, d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/4} \frac{R^3}{3} \left(-\cos\theta\right)_{\theta=0}^{\pi/2} d\varphi$$

$$= \int_{\varphi=0}^{\pi/4} \frac{R^3}{3} \, d\varphi = \frac{\pi R^3}{12}$$

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$$\iiint\limits_{\substack{x^2+y^2+z^2 \le R^2 \\ x,y,z \ge 0 \\ x \ge y}} x \, dV$$

$$= \int_{\varphi=0}^{\pi/4} \left(\int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{R} r \operatorname{sen}\theta \cos\varphi \cdot r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/4} \left(\int_{\theta=0}^{\pi/2} \frac{R^4}{4} \operatorname{sen}^2\theta \cos\varphi \, d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/4} \frac{R^4}{4} \frac{\pi}{4} \cos\varphi \, d\varphi$$

$$= \left. \frac{\pi R^4}{16} \operatorname{sen}\varphi \right]_{\varphi=0}^{\varphi=\pi/4}$$

$$= \frac{\pi R^4}{16} \frac{\sqrt{2}}{2} = \frac{\pi R^4 \sqrt{2}}{32}$$

$$\bar{x} = \frac{\pi R^4 \sqrt{2}/32}{\pi R^3/12} = \frac{3\sqrt{2}\, R}{8}$$

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$\boxed{D}$

$$\begin{array}{ccc|ccc} x & y & z & A & B & C \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 1 & 0 \\ 1 & 6 & 15 & 0 & 0 & 1 \end{array}$$

$$\xrightarrow[E3 \leftarrow E3 - E1]{E2 \leftarrow E2 - E1} \begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & -1 & 1 & 0 \\ 0 & 5 & 14 & -1 & 0 & 1 \end{array}$$

$$\xrightarrow{E2 \leftarrow E2/2} \begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1/2 & 1/2 & 0 \\ 0 & 5 & 14 & -1 & 0 & 1 \end{array}$$

$$\xrightarrow[E3 \leftarrow E3 - 5E2]{E1 \leftarrow E1 - E2} \begin{array}{cccccc} 1 & 0 & -1 & 3/2 & -1/2 & 0 \\ 0 & 1 & 2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 4 & 3/2 & -5/2 & 1 \end{array}$$

$$\xrightarrow{E3 \leftarrow E3/4} \begin{array}{cccccc} 1 & 0 & -1 & 3/2 & -1/2 & 0 \\ 0 & 1 & 2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 3/8 & -5/8 & 1/4 \end{array}$$

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$$
\begin{array}{l|ccc|ccc}
 & x & y & z & A & B & C \\
\hline
E1+E1+E3 & 1 & 0 & 0 & 15/8 & -9/8 & 1/4 \\
\xrightarrow{\hphantom{xxxxx}} & 0 & 1 & 0 & -5/4 & 7/4 & -1/2 \\
E2+E2-2E3 & 0 & 0 & 1 & 3/8 & -5/8 & 1/4
\end{array}
$$

$$
\rightarrow \quad
\begin{cases}
x = \frac{15}{8}A - \frac{9}{8}B + \frac{1}{4}C \\
y = -\frac{5}{4}A + \frac{7}{4}B - \frac{1}{2}C \\
z = \frac{3}{8}A - \frac{5}{8}B + \frac{1}{4}C
\end{cases}
$$

Controllo:

$$x+y+z = \left(\frac{15}{8} - \frac{5}{4} + \frac{3}{8}\right)A + \left(-\frac{9}{8} + \frac{7}{4} - \frac{5}{8}\right)B + \left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right)C = A$$

$$x+3y+5z = \left(\frac{15}{8} - \frac{15}{4} + \frac{15}{8}\right)A + \left(-\frac{9}{8} + \frac{21}{4} - \frac{25}{8}\right)B + \left(\frac{1}{4} - \frac{3}{2} + \frac{5}{4}\right)C = B$$

$$x+6y+15z = \left(\frac{15}{8} - \frac{30}{4} + \frac{45}{8}\right)A + \left(-\frac{9}{8} + \frac{42}{4} - \frac{75}{8}\right)B + \left(\frac{1}{4} - \frac{6}{2} + \frac{15}{4}\right)C = C$$