Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 22 febbraio 2019

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente. I compiti corretti saranno a disposizione mercoledì 6 marzo, verso le 10.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

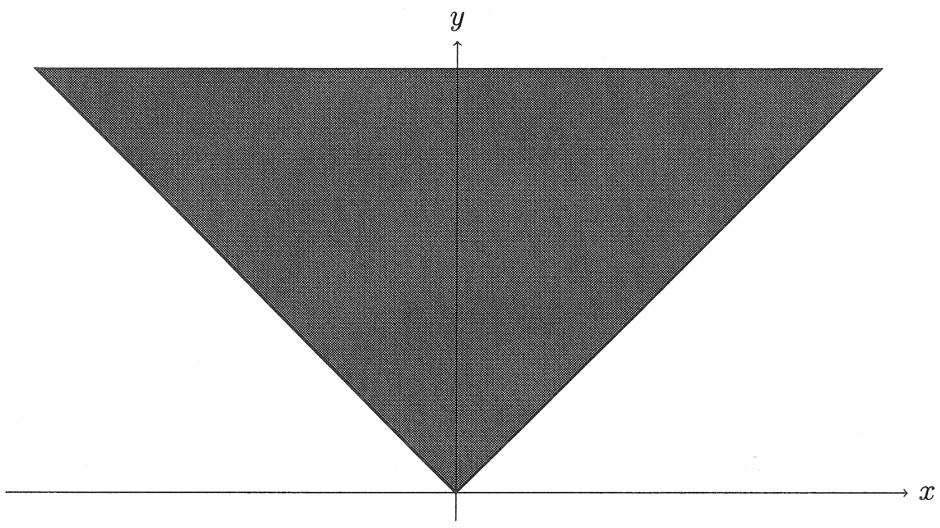
Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad z = z \qquad\qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

A. Calcolare

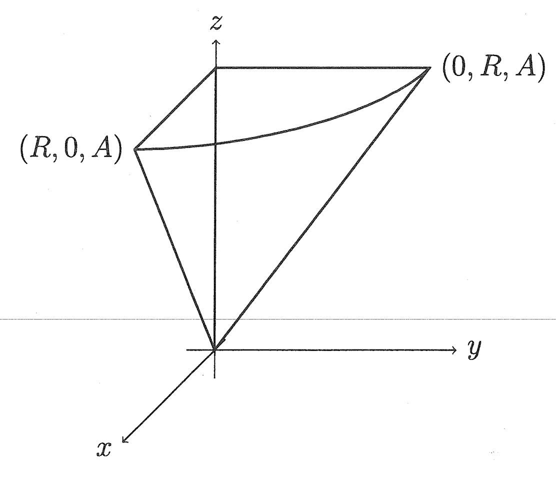
$$\int \frac{z^2}{(1-z^2)^3}\,dz$$

B. Sia A un parametro positivo. Usare le coordinate *polari* e calcolare la media di x^2y sul triangolo con vertici $(0,0)$, $(-A,A)$, e (A,A).



C. Siano A e R due parametri positivi. Usare le coordinate *cilindriche* e calcolare la media di xz sul quarto di un cono dato da:

$$0 \leq z \leq A \qquad x, y \geq 0 \qquad x^2 + y^2 \leq R^2z^2/A^2$$



D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema:

$$
\begin{array}{ccccl}
A & +B & +C & +D & = 1 \\
2A & +3B & +4C & +5D & = 0 \\
3A & +6B & +10C & +15D & = 1 \\
4A & +10B & +20C & +35D & = 16
\end{array}
$$

$\boxed{A}$

$$1-z^2=(1-z)(1+z)$$

$$(1-z^2)^3=(1-z)^3(1+z)^3$$

$$\frac{z^2}{(1-z^2)^3}=\frac{A}{1-z}+\frac{B}{(1-z)^2}+\frac{C}{(1-z)^3}$$

$$+\frac{D}{(1+z)}+\frac{E}{(1+z)^2}+\frac{F}{(1+z)^3}$$

$$z^2=A(1-z)^2(1+z)^3+B(1-z)(1+z)^3+C(1+z)^3$$

$$+D(1-z)^3(1+z)^2+E(1-z)^3(1+z)+F(1-z)^3$$

$$z=1 \rightsquigarrow 1^2=0+0+C(1+1)^3+0+0+0$$

$$\rightsquigarrow 1=8C \rightsquigarrow C=1/8$$

$$z=-1 \rightsquigarrow (-1)^2=0+0+0+0+0+F(1-(-1))^3$$

$$\rightsquigarrow 1=8F \rightsquigarrow F=1/8$$

$$z^2=A(1-z^2)^2(1+z)+B(1-z^2)(1+z)^2+\frac{1}{8}(1+z)^3$$

$$+D(1-z^2)^2(1-z)+E(1-z^2)(1-z)^2+\frac{1}{8}(1-z)^3$$

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$$= A(1-2z^2+z^4)(1+z) + D(1-2z^2+z^4)(1-z)$$
$$+ B(1-z^2)(1+2z+z^2) + E(1-z^2)(1-2z+z^2)$$
$$+ \frac{1}{8}(1+3z+3z^2+z^3) \quad + \frac{1}{8}(1-3z+3z^2-z^3)$$

$$= A(1-2z^2+z^4+z-2z^3+z^5)$$
$$+ D(1-2z^2+z^4-z+2z^3-z^5)$$
$$+ B(1+2z-2z^3+z^4)$$
$$+ E(1-2z+2z^3-z^4)$$
$$+ \frac{1}{8}(2+6z^2)$$

$$= (A+D+B+E+\frac{1}{4})$$
$$+(A-D+2B-2E)z$$
$$+(-2A-2D+\frac{3}{4})z^2$$
$$+(-2A+2D-2B+2E)z^3$$
$$+(B-E)z^4$$
$$+(A-D)z^5$$

$$\left.\begin{aligned} A+D+B+E+\tfrac{1}{4}&=0\\ A-D+2B-2E&=0\\ -2A-2D+\tfrac{3}{4}&=1\\ -2A+2D-2B+2E&=0 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} 2A+2B+\tfrac{1}{4}&=0\\ 0&=0\\ -4A+\tfrac{3}{4}&=1\\ 0&=0 \end{aligned}\right.$$

$$\left.\begin{aligned} B-E&=0\\ A-D&=0 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} E&=B\\ D&=A \end{aligned}\right\}$$

$$\longrightarrow -4A = 1-\frac{3}{4} = \frac{1}{4} \rightsquigarrow A = -1/16$$

$$2B = -2A - \frac{1}{4} = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8} \rightsquigarrow B = -\frac{1}{16}$$

$$\int\Big(-\frac{1}{16}\frac{1}{1-z} - \frac{1}{16}\frac{1}{(1-z)^2} + \frac{1}{8}\frac{1}{(1-z)^3}$$

$$-\frac{1}{16}\frac{1}{1+z} - \frac{1}{16}\frac{1}{(1+z)^2} + \frac{1}{8}\frac{1}{(1+z)^3}\Big)\,dz$$

$$= \frac{1}{16}\log(1-z) - \frac{1}{16}\frac{1}{1-z} + \frac{1}{16}\frac{1}{(1-z)^2}$$

$$-\frac{1}{16}\log(1+z) + \frac{1}{16}\frac{1}{1+z} - \frac{1}{16}\frac{1}{(1+z)^2}$$

B) $y \leq A \rightsquigarrow r \operatorname{sen}\theta \leq A \rightsquigarrow r \leq A/\operatorname{sen}\theta$

$0 \leq \theta \leq 2\pi$

$y \geq 0 \rightsquigarrow r \operatorname{sen}\theta \geq 0 \rightsquigarrow \operatorname{sen}\theta \geq 0$

$\rightsquigarrow 0 \leq \theta \leq \pi$

$y \geq x \rightsquigarrow r \operatorname{sen}\theta \geq r\cos\theta \rightarrow \operatorname{sen}\theta \geq \cos\theta$

$\rightsquigarrow \theta \geq \pi/4$

$y \geq -x \rightsquigarrow r \operatorname{sen}\theta \geq -r\cos\theta \rightarrow \operatorname{sen}\theta \geq -\cos\theta$

$\rightsquigarrow \theta \leq 3\pi/4$

$$\iint_R x^2 y \, dA$$

$$= \int_{\pi/4}^{3\pi/4} \left(\int_0^{A/\operatorname{sen}\theta} (r\cos\theta)^2 (r\operatorname{sen}\theta) r \, dr \right) d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \cos^2\theta \operatorname{sen}\theta \left(\int_0^{A/\operatorname{sen}\theta} r^4 \, dr \right) d\theta$$

$$= \frac{A^5}{5} \int_{\pi/4}^{3\pi/4} \cos^2\theta \operatorname{sen}\theta \left(\frac{1}{\operatorname{sen}\theta} \right)^5 d\theta$$

$$= \frac{A^5}{5} \int_{\pi/4}^{3\pi/4} \frac{\cos^2\theta}{\operatorname{sen}^4\theta}\, d\theta$$

$$= \frac{A^5}{5} \int_{\pi/4}^{3\pi/4} \frac{1-\operatorname{sen}^2\theta}{\operatorname{sen}^4\theta}\, d\theta$$

$$= \frac{A^5}{5} \int_{\pi/4}^{3\pi/4} \csc^4\theta - \csc^2\theta\, d\theta$$

$$\int_{\pi/4}^{3\pi/4} \csc^2\theta\, d\theta = -\operatorname{ctg}\theta\Big]_{\pi/4}^{3\pi/4}$$

$$= -(-1)-(-1) = 2$$

$$\int_{\pi/4}^{3\pi/4} \csc^4\theta\, d\theta$$

$$= -\frac{1}{3}\operatorname{ctg}\theta\csc^2\theta\Big]_{\theta=\pi/4}^{3\pi/4} + \frac{2}{3}\int_{\pi/4}^{3\pi/4}\csc^2\theta\, d\theta$$

$$= -\frac{1}{3}\left((-1)(\sqrt{2})^2 - 1(\sqrt{2})^2\right) + \frac{4}{3}$$

$$= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

$$\iint_{\mathcal{R}} x^2 y \, dA = \frac{A^5}{5}\left(\frac{8}{3}-2\right) = \frac{A^5}{5}\cdot\frac{2}{3}$$

$$= 2A^5/15$$

$$\overline{x^2 y} = \frac{\iint_{\mathcal{R}} x^2 y \, dA}{\iint_{\mathcal{R}} 1 \, dA} = \frac{2A^5/15}{(2A\cdot A)/2}$$

$$= \frac{2A^5/15}{A^2} = \frac{2}{15}A^3$$

$\boxed{C}$ $0 \le \theta \le 2\pi$

$x, y \ge 0 \rightsquigarrow r\cos\theta, r\,\text{sen}\,\theta \ge 0 \rightsquigarrow \cos\theta, \text{sen}\,\theta \ge 0$

$\rightsquigarrow 0 \le \theta \le \pi/2$

$x^2 + y^2 \le R^2 z^2/A^2 \rightsquigarrow r^2 \le R^2 z^2/A^2$

$\rightarrow r \le Rz/A$

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$$\iiint_G xz \, dV$$

$$= \int_{\theta=0}^{\pi/2} \left(\int_{z=0}^{A} \left(\int_{r=0}^{Rz/A} (r\cos\theta) z \, r \, dr \right) dz \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos\theta \left(\int_{z=0}^{A} z \left(\int_{r=0}^{Rz/A} r^2 \, dr \right) dz \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos\theta \left(\int_{z=0}^{A} z \, \frac{1}{3} \left(\frac{Rz}{A} \right)^3 dz \right) d\theta$$

$$= \frac{1}{3} \frac{R^3}{A^3} \int_{\theta=0}^{\pi/2} \cos\theta \left(\int_{z=0}^{A} z^4 \, dz \right) d\theta$$

$$= \frac{1}{3} \frac{R^3}{A^3} \frac{1}{5} A^5 \int_{\theta=0}^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{1}{15} R^3 A^2 \cdot \left[\sin\theta \right]_{\theta=0}^{\pi/2} = \frac{1}{15} R^3 A^2$$

$$\overline{xz} = \frac{\frac{1}{15} R^3 A^2}{\frac{1}{4} \cdot \frac{\pi}{3} R^2 A} = \cancel{\frac{1}{5\pi} RA} \quad \frac{4}{5\pi} RA$$

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$$
\begin{array}{ccccc}
A & B & C & D & = \\
1 & 1 & 1 & 1 & 1 \\
2 & 3 & 4 & 5 & 0 \\
3 & 6 & 10 & 15 & 1 \\
4 & 10 & 20 & 35 & 16
\end{array}
\xrightarrow[R4 \leftarrow R4 - 4R1]{R2 \leftarrow R2 - 2R1,\ R3 \leftarrow R3 - 3R1}
$$

$$
\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & -2 \\
0 & 3 & 7 & 12 & -2 \\
0 & 6 & 16 & 31 & 12
\end{array}
\xrightarrow[R4 \leftarrow R4 - 6R2]{R1 \leftarrow R1 - R2,\ R3 \leftarrow R3 - 3R2}
$$

$$
\begin{array}{ccccc}
1 & 0 & -1 & -2 & 3 \\
0 & 1 & 2 & 3 & -2 \\
0 & 0 & 1 & 3 & 4 \\
0 & 0 & 4 & 13 & 24
\end{array}
\xrightarrow[R4 \leftarrow R4 - 4R3]{R1 \leftarrow R1 + R3,\ R2 \leftarrow R2 - 2R3}
$$

$$
\begin{array}{ccccc}
1 & 0 & 0 & 1 & 7 \\
0 & 1 & 0 & -3 & -10 \\
0 & 0 & 1 & 3 & 4 \\
0 & 0 & 0 & 1 & 8
\end{array}
\xrightarrow[R3 \leftarrow R3 - 3R4]{R1 \leftarrow R1 - R4,\ R2 \leftarrow R2 + 3R4}
$$

$$
\begin{array}{ccccc}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 14 \\
0 & 0 & 1 & 0 & -20 \\
0 & 0 & 0 & 1 & 8
\end{array}
\rightsquigarrow
\boxed{\begin{array}{l} A = -1 \\ B = 14 \\ C = -20 \\ D = 8 \end{array}}
$$

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Controllo:

$-1+14-20+8=-21+22=1$ ✓

$2(-1)+3(14)+4(-20)+5(8)=-2+42-80+40$
$=-82+82=0$ ✓

$3(-1)+6(14)+10(-20)+15(8)$
$=-3+84-200+120=-203+204=1$ ✓

$4(-1)+10(14)+20(-20)+35(8)$
$=-4+140-400+280=-404+420$
$=16$ ✓