DA FOTOCOPIARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 13 settembre 2018

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente. I compiti corretti saranno a disposizione lunedì 17 settembre, alle 12.00, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Le formule per le coordinate sferiche sono:

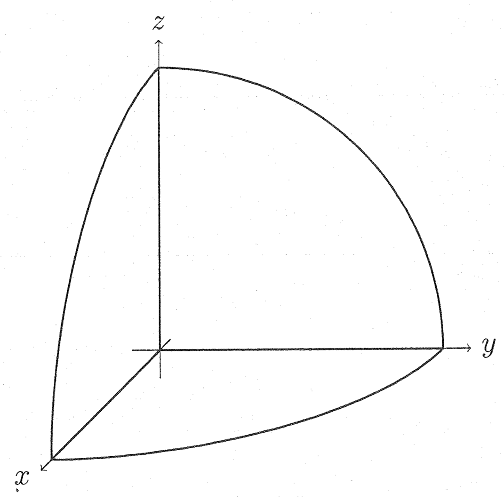
$$x = r\,\mathrm{sen}\,\theta\cos\phi \qquad y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\,\mathrm{sen}\,\theta\,dr\,d\theta\,d\phi$$

A. Calcolare

$$\int \frac{r^2}{(r^2+2)^2 - 4r^2}\,dr$$

B. Sia $R > 0$ un parametro costante. Usare le coordinate *sferiche* e calcolare la media di xyz sul volume definito da

$$x, y, z \geq 0 \qquad\qquad x^2 + y^2 + z^2 \leq R^2$$



C. Lo stesso dell'esercizio precedente, ma con le coordinate *cilindriche.*

D. Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio delle soluzioni del sistema:

$$\begin{aligned} p + q + r &= 3 \\ q + r + s &= 3 \\ r + s + t &= 3 \\ s + t + u &= 3 \\ t + u + p &= 3 \\ u + p + q &= 3 \end{aligned}$$

A $(n^2+2)^2-4n^2=(n^2+2+2n)(n^2+2-2n)$

$n^2+2n+2 \rightsquigarrow B^2-4AC=4-4\cdot1\cdot2=-4$
< 0

$n^2-2n+2 \rightsquigarrow B^2-4AC=4-4\cdot1\cdot2=-4$
< 0

$$\frac{n^2}{(n^2+2n+2)(n^2-2n+2)}=$$

$$A\,\frac{2n+2}{n^2+2n+2}+B\,\frac{1}{n^2+2n+2}$$

$$+\,C\,\frac{2n-2}{n^2-2n+2}+D\,\frac{1}{n^2-2n+2}$$

$\longrightarrow$

$$n^2=A(2n+2)(n^2-2n+2)$$
$$+\,B(n^2-2n+2)$$
$$+\,C(2n-2)(n^2+2n+2)$$
$$+\,D(n^2+2n+2)$$

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$$\begin{aligned} &= A(2n^3 - 2n^2 + 4) \\ &+ B(n^2 - 2n + 2) \\ &+ C(2n^3 + 2n^2 - 4) \\ &+ D(n^2 + 2n + 2) \end{aligned}$$

A	B	C	D	=
2	0	2	0	0
-2	1	2	1	1
0	-2	0	2	0
4	2	-4	2	0

$$\xrightarrow[\substack{R4 \leftarrow R4/2 \\ R3 \leftarrow R3/2}]{R1 \leftarrow R1/2} \begin{matrix} 1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 2 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 1 & 0 \end{matrix}$$

$$\xrightarrow[R4 \leftarrow R4 - 2R1]{R2 \leftarrow R2 + 2R1} \begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 1 & 0 \end{matrix}$$

$$\xrightarrow[R4 \leftarrow R4 - R2]{R3 \leftarrow R3 + R2} \begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & -8 & 0 & -1 \end{matrix}$$

$$\xrightarrow{R3 \leftarrow R3/4} \begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 1/2 & 1/4 \\ 0 & 0 & -8 & 0 & -1 \end{matrix}$$

$$\xrightarrow[R4 \leftarrow R4 + 8R3]{R1 \leftarrow R1 - R3,\ R2 \leftarrow R2 - 4R3} \begin{matrix} 1 & 0 & 0 & -1/2 & -1/4 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/4 \\ 0 & 0 & 0 & 4 & 1 \end{matrix}$$

$$\xrightarrow{R4 \leftarrow R4/4} \begin{matrix} 1 & 0 & 0 & -1/2 & -1/4 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1 & 1/4 \end{matrix}$$

$$\begin{matrix} R1 \leftarrow R1 + \frac{1}{2}R4 \\ R2 \leftarrow R2 + R4 \\ R3 \leftarrow R3 - \frac{1}{2}R4 \end{matrix} \quad \begin{matrix} 1 & 0 & 0 & 0 & -1/8 \\ 0 & 1 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 1/8 \\ 0 & 0 & 0 & 1 & 1/4 \end{matrix}$$

$$\int \frac{x^2}{(x^2+2x+2)(x^2-2x+2)}\,dx$$

$$= \int -\frac{1}{8}\,\frac{2x+2}{x^2+2x+2} + \frac{1}{4}\,\frac{1}{x^2+2x+2}$$

$$+ \frac{1}{8}\,\frac{2x-2}{x^2-2x+2} + \frac{1}{4}\,\frac{1}{x^2-2x+2}\,dx$$

$$= -\frac{1}{8}\log(x^2+2x+2)$$

$$+ \frac{1}{4}\,\frac{2}{\sqrt{4}}\,\operatorname{arctg}\left(\frac{2x+2}{\sqrt{4}}\right)$$

$$+ \frac{1}{8}\log(x^2-2x+2)$$

$$+ \frac{1}{4}\,\frac{2}{\sqrt{4}}\,\operatorname{arctg}\left(\frac{2x-2}{\sqrt{4}}\right)$$

$$= \boxed{\begin{aligned} &-\frac{1}{8}\log(x^2+2x+2) \\ &+ \frac{1}{4}\operatorname{arctg}(x+1) \\ &+ \frac{1}{8}\log(x^2-2x+2) \\ &+ \frac{1}{4}\operatorname{arctg}(x-1) \end{aligned}}$$

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(A) $\iiint_{V} 1 \, dV = \frac{1}{8} \cdot \frac{4\pi R^3}{3} = \frac{\pi R^3}{6}$

$x, y \geq 0 \to r \operatorname{sen}\theta \cos\varphi, \; r \operatorname{sen}\theta \operatorname{sen}\varphi \geq 0$

$\to \cos\varphi, \operatorname{sen}\varphi \geq 0$

$\to 0 \leq \varphi \leq \pi/2$

$z \geq 0 \to r\cos\theta \geq 0 \to 0 \leq \theta \leq \frac{\pi}{2}$

$x^2+y^2+z^2 \leq R^2 \to r^2 \leq R^2 \to 0 \leq r \leq R$

$$\iiint_{V} xyz \, dV$$

$$= \int_{\varphi=0}^{\pi/2} \left(\int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{R} (r \operatorname{sen}\theta \cos\varphi)(r \operatorname{sen}\theta \operatorname{sen}\varphi)(r\cos\theta) \cdot r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$\int_{\varphi=0}^{\pi/2} \operatorname{sen}\varphi \cos\varphi \left(\int_{\theta=0}^{\pi/2} \operatorname{sen}^3\theta \cos\theta \left(\int_{r=0}^{R} r^5 \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \operatorname{sen}\varphi \cos\varphi \left(\int_{\theta=0}^{\pi/2} \operatorname{sen}^3\theta \cos\theta \, \frac{R^6}{6} \, d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \operatorname{sen}\varphi \cos\varphi \left(\left[\frac{\operatorname{sen}^4\theta}{4} \right]_{\theta=0}^{\pi/2} \frac{R^6}{6} \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \operatorname{sen}\varphi \cos\varphi \, \frac{R^6}{24} \, d\varphi$$

$$= \left[\frac{\operatorname{sen}^2\varphi}{2} \right]_{\varphi=0}^{\pi/2} \frac{R^6}{24} = \frac{1}{2} \frac{R^6}{24}$$

$$= \frac{R^6}{48}$$

$$\overline{xyz} = \frac{R^6/48}{\pi R^3/6} = \boxed{\frac{R^3}{8\pi}}$$

(c) $x, y \geq 0 \rightsquigarrow r\cos\theta,\ r\,\mathrm{sen}\,\theta \geq 0$

$\rightarrow \cos\theta,\ \mathrm{sen}\,\theta \geq 0 \rightsquigarrow 0 \leq \theta \leq \frac{\pi}{2}$

$x^2+y^2+z^2 \leq R^2 \rightsquigarrow r^2+z^2 \leq R^2$

$\rightsquigarrow r \leq \sqrt{R^2-z^2}$

e anche $z \leq R$.

$$\iiint_V xyz\, dV$$

$$= \int_{\theta=0}^{\pi/2} \left(\int_{z=0}^{R} \left(\int_{r=0}^{\sqrt{R^2-z^2}} r\,\mathrm{sen}\,\theta \cdot r\cos\theta \cdot z\ r\, dr \right) dz \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \mathrm{sen}\,\theta \cos\theta \left(\int_{z=0}^{R} z \left(\int_{r=0}^{\sqrt{R^2-z^2}} r^3\, dr \right) dz \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta \cos\theta \left(\int_{z=0}^{R} z \, \frac{\left(\sqrt{R^2-z^2}\right)^4}{4} \, dz \right) d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta \cos\theta \left(\int_{z=0}^{R} z \left(R^2-z^2\right)^2 dz \right) d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta \cos\theta \left(\int_{z=0}^{R} zR^4 - 2z^3R^2 + z^5 \, dz \right) d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta \cos\theta \left(\frac{R^6}{2} - \frac{2R^6}{4} + \frac{R^6}{6} \right) d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta \cos\theta \, \frac{R^6}{6} \, d\theta$$

$$= \frac{R^6}{24} \left. \frac{1}{2} \operatorname{sen}^2\theta \right]_{\theta=0}^{\pi/2} = \frac{R^6}{48}$$

$$\overline{xyz} = \frac{R^6/48}{\pi R^3/6} = \frac{R^3}{8\pi}$$

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$$
\begin{array}{cccccccc}
p & q & r & s & t & u & = & \\
1 & 1 & 1 & 0 & 0 & 0 & & 3 \\
0 & 1 & 1 & 1 & 0 & 0 & & 3 \\
0 & 0 & 1 & 1 & 1 & 0 & & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & & 3 \\
1 & 0 & 0 & 0 & 1 & 1 & & 3 \\
1 & 1 & 0 & 0 & 0 & 1 & & 3
\end{array}
$$

$$
\xrightarrow[E6 \leftarrow E6 - E1]{E5 \leftarrow E5 - E1}
\begin{array}{ccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 3 \\
0 & 1 & 1 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 1 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 \\
0 & -1 & -1 & 0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0
\end{array}
$$

$$
\xrightarrow[E5 \leftarrow E5 + E2]{E1 \leftarrow E1 - E2}
\begin{array}{ccccccc}
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 1 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & 1 & 1 & 1 & 3 \\
0 & 0 & -1 & 0 & 0 & 1 & 0
\end{array}
$$

$$\xrightarrow[E6 \leftarrow E6 + E3]{E2 \leftarrow E2 - E3} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow[\substack{E5 \leftarrow E5 - E4 \\ E6 \leftarrow E6 - E4}]{\substack{E1 \leftarrow E1 + E4 \\ E3 \leftarrow E3 - E4}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} p + t + u = 3 \\ q - t = 0 \\ r - u = 0 \\ s + t + u = 3 \end{cases} \longrightarrow \boxed{\begin{aligned} p &= 3 - t - u \\ q &= t \\ r &= u \\ s &= 3 - t - u \\ t &= t \\ u &= u \end{aligned}}$$