Da fotocopiare

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 28 settembre 2018

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente. I compiti corretti saranno a disposizione venerdì 5 ottobre, alle 12.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad dx\,dy = r\,dr\,d\theta$$

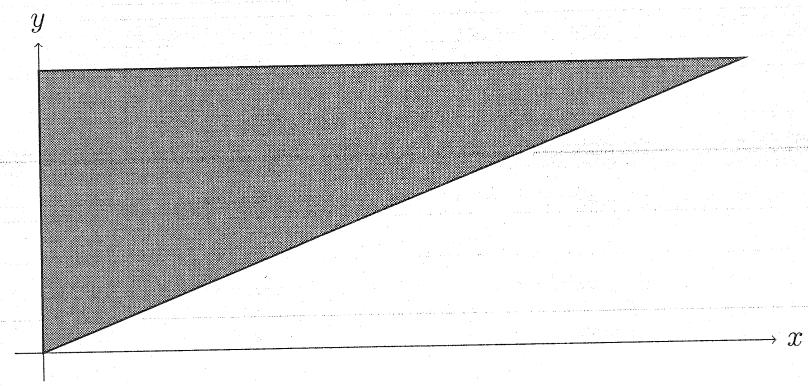
Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

A. Usare la sostituzione $t = s^5$ e calcolare

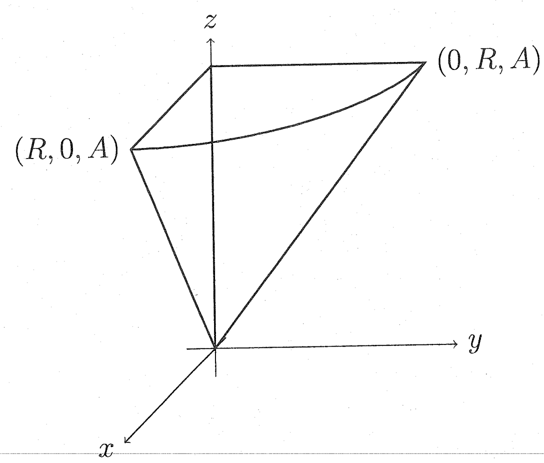
$$\int \frac{s^4 + s^9 + s^{14}}{s^{10} - 5s^5 + 6}\,ds$$

B. Siano A e B due parametri positivi. Usare le coordinate *polari* e calcolare la media di $\sqrt{x^2+y^2}$ sul triangolo con vertici $(0,0)$, $(0,A)$, e (B,A).



C. Siano A e R due parametri positivi. Usare le coordinate *cilindriche* e calcolare la media di xy sul quarto di un ~~cilindro~~ cono dato da:

$$0 \le z \le A \qquad x, y \ge 0 \qquad x^2+y^2 \le R^2z^2/A^2$$



D. Usare il metodo di Gauss–Jordan ed esprimere x, y, e z in termini di A, B, e C.

$$
\begin{aligned}
x + y/3 + z/5 &= A \\
x/2 + y/4 + z/6 &= B \\
x/3 + y/5 + z/7 &= C
\end{aligned}
$$

$\boxed{A}$

$t = s^5$
$dt = 5s^4 ds$

$$\int \frac{s^4 + s^9 + s^{14}}{s^{10} - 5s^5 + 6}\, ds$$

$$= \frac{1}{5} \int \frac{1 + s^5 + s^{10}}{s^{10} - 5s^5 + 6}\, 5 \cdot s^4\, ds$$

$$= \frac{1}{5} \int \frac{1 + t + t^2}{t^2 - 5t + 6}\, dt$$

$$\begin{array}{r|l} t^2 + t + 1 & t^2 - 5t + 6 \\ \hline t^2 - 5t + 6 & 1 \\ \hline 6t - 5 & \end{array}$$

$\rightsquigarrow$ $t^2 + t + 1 = (t^2 - 5t + 6) + (6t - 5)$

$$\rightsquigarrow \frac{t^2 + t + 1}{t^2 - 5t + 6} = 1 + \frac{6t - 5}{t^2 - 5t + 6}$$

$$\frac{6t - 5}{t^2 - 5t + 6} = \frac{6t - 5}{(t-2)(t-3)} = \frac{A}{t-2} + \frac{B}{t-3}$$

$$\rightsquigarrow 6t - 5 = A(t-3) + B(t-2)$$

$t = 2$ $\quad 6 \cdot 2 - 5 = A(2-3) + 0 = -A \rightsquigarrow A = -7$

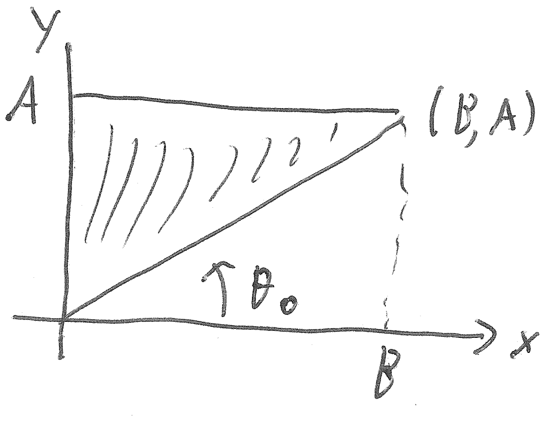
$t = 3$ $\quad 6 \cdot 3 - 5 = 0 + B(3-2) = B \rightsquigarrow B = 13$

$$\frac{1}{5}\int 1-7\frac{1}{t-2}+13\frac{1}{t-3}\,dt$$

$$=\frac{t}{5}-\frac{7}{5}\log(t-2)+\frac{13}{5}\log(t-3)$$

$$=\frac{5^5}{5}-\frac{7}{5}\log(5^5-2)+\frac{13}{5}\log(5^5-3)$$

$\boxed{B}$



$[\ tg\,\theta_0 = A/B$

$ctg\,\theta_0 = B/A$

$csc\,\theta_0 = \sqrt{A^2+B^2}/A\ ]$

$y \le A \rightsquigarrow r\,sen\,\theta \le A \rightsquigarrow r \le A\csc\theta$

$x \ge 0 \rightsquigarrow r\cos\theta \ge 0 \rightsquigarrow \cos\theta \ge 0$

$\rightsquigarrow -\pi/2 \le \theta \le \pi/2$

$y \ge \frac{A}{B}x \rightsquigarrow r\,sen\,\theta \ge \frac{A}{B} r\cos\theta$

$\rightsquigarrow tg\,\theta \ge \frac{A}{B} \rightsquigarrow \theta \ge arctg\left(\frac{A}{B}\right) = \theta_0$

$$\iint_{\text{triangolo}} 1 \, dA = \frac{1}{2} A B$$

$$\iint_{\text{triangolo}} r \, dA = \int_{\theta=\theta_0}^{\pi/2} \left(\int_{r=0}^{A \csc\theta} r \cdot r \, dr \right) d\theta$$

$$= \int_{\theta=\theta_0}^{\pi/2} \frac{1}{3} (A \csc\theta)^3 \, d\theta$$

$$= \int_{\theta=\theta_0}^{\pi/2} \frac{1}{3} A^3 \csc^3\theta \, d\theta$$

$$= \frac{1}{3} A^3 \left[-\frac{1}{2} \csc\theta \, \mathrm{ctg}\,\theta - \frac{1}{2} \log(\mathrm{ctg}\,\theta + \csc\theta) \right]_{\theta=\theta_0}^{\pi/2}$$

$$= -\frac{1}{6} A^3 \Big[+ \cancel{\tfrac{1}{2}} \cdot 1 \cdot 0 + \cancel{\tfrac{1}{2}} \log(0+1)$$

$$- \cancel{\tfrac{1}{2}} \frac{B}{A} \frac{\sqrt{A^2+B^2}}{A} - \log\left(\frac{B}{A} + \frac{\sqrt{B^2+A^2}}{A} \right) \Big]$$

$$= +\frac{1}{6} B A \sqrt{A^2+B^2} + \frac{1}{6} A^3 \log\left(\frac{B + \sqrt{A^2+B^2}}{A} \right)$$

$$\overline{\sqrt{x^2+y^2}} = \overline{r}$$

$$= \frac{\frac{1}{6} BA\sqrt{A^2+B^2} + \frac{1}{6} A^3 \log\left(\frac{B+\sqrt{A^2+B^2}}{A}\right)}{\frac{1}{2} AB}$$

$$= \boxed{\frac{1}{3}\sqrt{A^2+B^2} + \frac{1}{3}\frac{A^2}{B}\log\left(\frac{B+\sqrt{A^2+B^2}}{A}\right)}$$

$\boxed{C}$

$x, y \geq 0 \rightsquigarrow r\cos\theta,\ r\,\mathrm{sen}\,\theta \geq 0$

$\rightsquigarrow \cos\theta, \mathrm{sen}\,\theta \geq 0 \rightsquigarrow 0 \leq \theta \leq \pi/2$

$x^2+y^2 \leq R^2 z^2/A^2 \rightsquigarrow r^2 \leq R^2 z^2/A^2$

$\rightsquigarrow r \leq Rz/A$

$$\iiint_{\text{volume}} 1 \, dV = \frac{1}{4}\cdot\frac{1}{3}\pi R^2 A = \frac{\pi R^2 A}{12}$$

p. 5/8

$$\iiint_{\text{volume}} xy \, dV$$

$$= \int_{\theta=0}^{\pi/2} \left(\int_{z=0}^{A} \left(\int_{r=0}^{Rz/A} (r \operatorname{sen}\theta)(r\cos\theta) \, r \, dr \right) dz \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\int_{z=0}^{A} \frac{1}{4} \left(\frac{Rz}{A} \right)^4 \operatorname{sen}\theta \cos\theta \, dz \right) d\theta$$

$$= \frac{R^4}{4A^4} \int_{\theta=0}^{\pi/2} \left(\int_{z=0}^{A} z^4 \operatorname{sen}\theta \cos\theta \right) d\theta$$

$$= \frac{R^4}{4A^4} \int_{\theta=0}^{\pi/2} \frac{A^5}{5} \operatorname{sen}\theta \cos\theta \, d\theta$$

$$= \frac{R^4 A}{20} \left. \frac{\operatorname{sen}^2\theta}{2} \right]_{\theta=0}^{\theta=\pi/2} = \frac{R^4 A}{40}$$

$$\overline{xy} = \frac{R^4 A / 40}{\pi R^2 A / 12} = \frac{12 R^2}{40\pi} = \frac{3R^2}{10\pi}$$

p. 6 / 8

x	y	z	$=$	A	B	C
1	$1/3$	$1/5$		1	0	0
$1/2$	$1/4$	$1/6$		0	1	0
$1/3$	$1/5$	$1/7$		0	0	1

$$\xrightarrow[E3 \leftarrow E3 - \frac{1}{3}E1]{E2 \leftarrow E2 - \frac{1}{2}E1}$$

$$\begin{array}{cccccc} 1 & 1/3 & 1/5 & 1 & 0 & 0 \\ 0 & 1/12 & 1/15 & -1/2 & 1 & 0 \\ 0 & 4/45 & 8/105 & -1/3 & 0 & 1 \end{array}$$

$$\xrightarrow{E2 \leftarrow 12E2} \begin{array}{cccccc} 1 & 1/3 & 1/5 & 1 & 0 & 0 \\ 0 & 1 & 4/5 & -6 & 12 & 0 \\ 0 & 4/45 & 8/105 & -1/3 & 0 & 1 \end{array}$$

$$\xrightarrow[E3 \leftarrow E3 - \frac{4}{45}E2]{E1 \leftarrow E1 - \frac{1}{3}E2}$$

p. 7/8

$$\begin{matrix} 1 & 0 & -1/15 & 3 & -4 & 0 \\ 0 & 1 & 4/5 & -6 & 12 & 0 \\ 0 & 0 & 8/1575 & 1/5 & -16/15 & 1 \end{matrix}$$

$$\xrightarrow{E3 \leftarrow E3 \cdot 1575/8}$$

$$\begin{matrix} 1 & 0 & -1/15 & 3 & -4 & 0 \\ 0 & 1 & 4/5 & -6 & 12 & 0 \\ 0 & 0 & 1 & 315/8 & -210 & 1575/8 \end{matrix}$$

$$\xrightarrow[E2 \leftarrow E2 - \frac{4}{5} E3]{E1 \leftarrow E1 + \frac{1}{15} \cdot E3}$$

$$\begin{matrix} 1 & 0 & 0 & 45/8 & -18 & 105/8 \\ 0 & 1 & 0 & -75/2 & 180 & -315/2 \\ 0 & 0 & 1 & 315/8 & -210 & 1575/8 \end{matrix}$$

$$x = \frac{45}{8}A - 18B + \frac{105}{8}C$$

$$y = \frac{-75}{2}A + 180B - \frac{315}{2}C$$

$$z = \frac{315}{8}A - 210B + \frac{1575}{8}C$$

Controllo:

$$x + \frac{1}{3}y + \frac{1}{5}z = \frac{45}{8}A - 18B + \frac{105}{8}C$$
$$+ \frac{1}{3}\left(\frac{-75}{2}A + 180B - \frac{315}{2}\right)C$$
$$+ \frac{1}{5}\left(\frac{315}{8}A - 210B + \frac{1575}{8}\right)C$$

$$= \left(\frac{45}{8} - \frac{25}{2} + \frac{63}{8}\right)A + (-18 + 60 - 42)B$$
$$+ \left(\frac{105}{8} - \frac{105}{2} + \frac{315}{8}\right)C$$

$$= \frac{45 - 100 + 63}{8}A + 0 \cdot B + \frac{105 - 420 + 315}{8}C$$

$$= \frac{8}{8}A + 0 \cdot B + 0 \cdot C = A \; ✓$$

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