Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 14 luglio 2017

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato. È a disposizione anche una tavola per la distribuzione normale standardizzata.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Le formule per le coordinate sferiche sono:

$$x = r\,\mathrm{sen}\,\theta\cos\phi \qquad y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\,\mathrm{sen}\,\theta\,dr\,d\theta\,d\phi$$

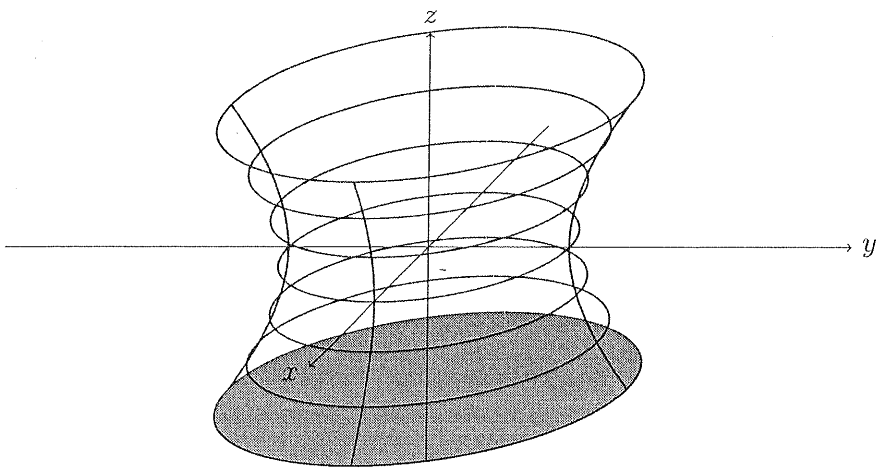
A. Calcolare

$$\int \frac{y^2 - y + 1}{(y^2 + y + 1)^2}\,dy$$

B. Sia $\mathcal{I}$ il volume (pezzo di un iperboloide) definito da:

$$-1 \leq z \leq 1 \qquad\qquad x^2 + y^2 \leq z^2 + 1$$

Usare le coordinate cilindriche e calcolare la media di x^2 su $\mathcal{I}$.



C. Siano $R > S > 0$ e A tre costanti. Sia $\mathcal{V}$ il volume definito da

$$S^2 \leq x^2 + y^2 + z^2 \leq R^2 \qquad\qquad z \leq 0 \qquad\qquad x^2 + y^2 \leq z^2$$

Usare le coordinate sferiche e calcolare

$$\iiint_{\mathcal{V}} (x^2 + y^2 + z^2)^A \, dV$$

D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema:

$$a + b = 10 \qquad a + c = 12 \qquad a + d = 15$$
$$c + d = 20 \qquad b + d = 18 \qquad b + c = 15$$

A $$\frac{y^2-y+1}{(y^2+y+1)^2} = \frac{A}{y^2+y+1} + \frac{B(2y+1)}{y^2+y+1}$$

$$+ \frac{C}{(y^2+y+1)^2} + \frac{D(2y+1)}{(y^2+y+1)^2}$$

$$y^2-y+1 = A(y^2+y+1) + B(2y+1)(y^2+y+1)$$
$$+ C + D(2y+1)$$

$$\begin{aligned} = &(2B)y^3 \\ &+(A+2B+B)y^2 \\ &+(A+2B+B+2D)y \\ &+(A+B+C+D) \end{aligned}$$

$$\rightarrow \begin{aligned} B &= 0 \\ A+3B &= 1 \\ A+3B+2D &= -1 \\ A+B+C+D &= 1 \end{aligned} \qquad \rightarrow \begin{aligned} B &= 0 \\ A &= 1 \\ 1+2D &= -1 \\ 1+C+D &= 1 \end{aligned}$$

$$\rightarrow B=0,\ A=1,\ D=-1,\ C=1$$

$$\int \frac{2y+1}{(y^2+y+1)^2}\,dy = -\frac{1}{y^2+y+1}$$

$$\int \frac{1}{y^2+y+1}\,dy = \int \frac{1}{ay^2+by+c}\,dy$$

$$= \frac{2}{\sqrt{4ac-b^2}}\,\operatorname{arctg}\left(\frac{2ay+b}{\sqrt{4ac-b^2}}\right)$$

$$= \frac{2}{\sqrt{3}}\,\operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right)$$

$$\int \frac{1}{(y^2+y+1)^2}\,dy = \int \frac{1}{\left((y+\frac{1}{2})^2+\frac{3}{4}\right)^2}\,dy$$

$$= \int \frac{1}{\left((y+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2\right)^2}\,dy$$

Si usa $\quad y+\frac{1}{2} = \frac{\sqrt{3}}{2}\operatorname{tg} t$

$$dy = \frac{\sqrt{3}}{2}\sec^2 t\,dt$$

$$t = \operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right)$$

$$\int \frac{1}{\left(\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right)^2}\,dy$$

$$= \int \frac{1}{\left(\left(\frac{\sqrt{3}}{2}\,tg\,t\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right)^2}\,\frac{\sqrt{3}}{2}\sec^2 t\,dt$$

$$= \int \frac{1}{\left(\frac{3}{4}\left(tg^2 t+1\right)\right)^2}\,\frac{\sqrt{3}}{2}\sec^2 t\,dt$$

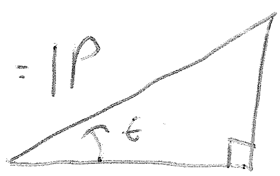
$$= \int \frac{16}{9}\,\frac{\sqrt{3}}{2}\,\frac{\sec^2 t}{\sec^4 t}\,dt$$

$$= \frac{8\sqrt{3}}{9}\int \cos^2 t\,dt$$

$$= \frac{4\sqrt{3}}{9}\left(\operatorname{sen} t\cos t + t\right)$$

$$t = \operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right)$$

$$\sqrt{(2y+1)^2+(\sqrt{3})^2} = IP \qquad 2y+1 = OP$$



$$= \sqrt{4y^2+4y+4}$$

$$= 2\sqrt{y^2+y+1} \qquad \sqrt{3} = AD$$

$$\operatorname{sen} t = \frac{OP}{1P} = \frac{2y+1}{2\sqrt{y^2+y+1}}$$

$$\cos t = \frac{A1}{1P} = \frac{\sqrt{3}}{2\sqrt{y^2+y+1}}$$

$$\frac{4\sqrt{3}}{9}\left(\operatorname{sen} t \cos t + t\right)$$

$$= \frac{4\sqrt{3}}{9}\left(\frac{(2y+1)\sqrt{3}}{4(y^2+y+1)} + \operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{3}\,\frac{2y+1}{y^2+y+1} + \frac{4\sqrt{3}}{9}\operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right)$$

$$\int \frac{y^2-y+1}{(y^2+y+1)^2}\,dy = \int \frac{1}{y^2+y+1} + \frac{1}{(y^2+y+1)^2} - \frac{2y+1}{(y^2+y+1)^2}\,dy$$

$$= \frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right) + \frac{1}{3}\,\frac{2y+1}{y^2+y+1}$$

$$+ \frac{4\sqrt{3}}{9}\operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right) + \frac{1}{y^2+y+1}$$

$$= \frac{10}{3\sqrt{3}} \operatorname{arctg}\left(\frac{2y+1}{\sqrt{3}}\right) + \frac{2y+4}{3(y^2+y+1)}$$

B

$$-1 \leq z \leq 1 \qquad r^2 \leq z^2+1$$

$$\int_{\theta=0}^{\theta=2\pi} \left(\int_{z=-1}^{z=1} \left(\int_{r=0}^{r=\sqrt{z^2+1}} 1 \cdot r \, dr \right) dz \right) d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\int_{z=-1}^{z=1} \frac{z^2+1}{2} \, dz \right) d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} \left. \frac{z^3}{3} + z \right]_{z=-1}^{z=1} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} \frac{2}{3} + 2 \, d\theta$$

$$= \frac{1}{2}(2\pi) \cdot \frac{8}{3} = \frac{8}{3}\pi$$

$$\int_{\theta=0}^{\theta=2\pi} \left(\int_{z=-1}^{z=1} \left(\int_{r=0}^{r=\sqrt{z^2+1}} (r\cos\theta)^2 \, r \, dr \right) d\theta \right) dz$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\int_{z=-1}^{z=1} \left(\int_{r=0}^{r=\sqrt{z^2+1}} \cos^2\theta \cdot r^3 \, dr \right) d\theta \right) dz$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\int_{z=-1}^{z=1} \cos^2\theta \cdot \frac{1}{4} (z^2+1)^2 \, dz \right) d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\theta=2\pi} \left(\int_{z=-1}^{z=1} \cos^2\theta \left(z^4 + 2z^2 + 1 \right) dz \right) d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \left(\frac{z^5}{5} + \frac{2z^3}{3} + z \right)_{z=-1}^{z=+1} d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \left(\frac{2}{5} + \frac{4}{3} + 2 \right) d\theta$$

$$= \frac{1}{4} \cdot \pi \cdot \frac{56}{15} = \frac{14}{15}\pi$$

p. 7/11

$$\overline{x^2} = \frac{\iiint_{\mathcal{D}} x^2 \, dV}{\iiint_{\mathcal{D}} 1 \, dV} = \frac{\frac{14}{15}\pi}{\frac{8}{3}\pi}$$

$$= \frac{14}{15} \cdot \frac{3}{8} = \frac{7}{5} \cdot \frac{1}{4} = \frac{7}{20}$$

(c) $S^2 \leq x^2+y^2+z^2 \leq R^2$ e $r \geq 0$

$\hookrightarrow S^2 \leq r^2 \leq R^2$

$\hookrightarrow S \leq r \leq R$

$z \leq 0$ e $0 \leq \theta \leq \pi$

$\hookrightarrow r\cos\theta \leq 0$

$\hookrightarrow \cos\theta \leq 0 \longrightarrow \frac{\pi}{2} \leq \theta \leq \pi$

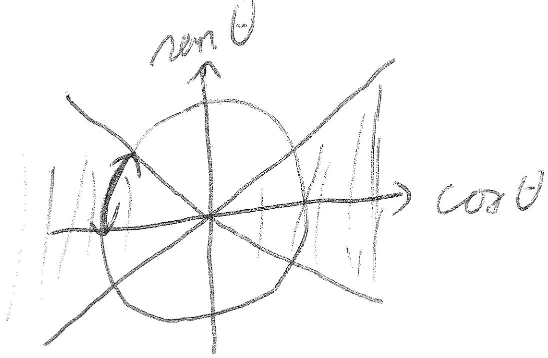
$x^2+y^2 \leq z^2$

$\hookrightarrow r^2 \operatorname{sen}^2\theta \cos^2\varphi + r^2 \operatorname{sen}^2\theta \operatorname{sen}^2\varphi \leq r^2\cos^2\theta$

$\hookrightarrow \operatorname{sen}^2\theta(\cos^2\varphi + \operatorname{sen}^2\varphi) \leq \cos^2\theta$

$\hookrightarrow \operatorname{sen}^2\theta \leq \cos^2\theta \longrightarrow |\operatorname{sen}\theta| \leq |\cos\theta|$

$|\operatorname{sen}\theta| \leq |\cos\theta|$



$$\frac{3\pi}{4} \leq \theta \leq \pi$$

$$\iiint_{V} (x^2+y^2+z^2)^A \, dV$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(\int_{\theta=3\pi/4}^{\theta=\pi} \left(\int_{r=S}^{r=R} (r^2)^A r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(\int_{\theta=\frac{3\pi}{4}}^{\theta=\pi} \left(\int_{r=S}^{r=R} \operatorname{sen}\theta \, r^{2A+2} \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(\int_{\theta=\frac{3\pi}{4}}^{\theta=\pi} \operatorname{sen}\theta \left(\frac{R^{2A+3} - S^{2A+3}}{2A+3} \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \Big[-\cos\theta \Big]_{\theta=\frac{3\pi}{4}}^{\theta=\pi} \left(\frac{R^{2A+3} - S^{2A+3}}{2A+3} \right) d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(1-\frac{\sqrt{2}}{2}\right)\left(\frac{R^{2A+3}-S^{2A+3}}{2A+3}\right) d\varphi$$

$$= \pi\left(2-\sqrt{2}\right)\left(\frac{R^{2A+3}-S^{2A+3}}{2A+3}\right)$$

D

a	b	c	d	= cost.
1	1	0	0	10
0	0	1	1	20
1	0	1	0	12
0	1	0	1	18
1	0	0	1	15
0	1	1	0	15

$E3 \leftarrow E3 - E1$
$E5 \leftarrow E5 - E1$

$$\begin{array}{ccccc} 1 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 & 20 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 18 \\ 0 & -1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 0 & 15 \end{array} \xrightarrow{E2 \leftrightarrow E3} \begin{array}{ccccc} 1 & 1 & 0 & 0 & 10 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 20 \\ 0 & 1 & 0 & 1 & 18 \\ 0 & -1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 0 & 15 \end{array}$$

$$\xrightarrow{E2 \leftarrow -E2} \begin{pmatrix} 1 & 1 & 0 & 0 & 10 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 20 \\ 0 & 1 & 0 & 1 & 18 \\ 0 & -1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 0 & 15 \end{pmatrix}$$

$$\xrightarrow[\substack{E5 \leftarrow E5+E2 \\ E6 \leftarrow E6-E2}]{\substack{E1 \leftarrow E1-E2 \\ E4 \leftarrow E4-E2}} \begin{pmatrix} 1 & 0 & 1 & 0 & 12 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 20 \\ 0 & 0 & 1 & 1 & 20 \\ 0 & 0 & -1 & 1 & 3 \\ 0 & 0 & 2 & 0 & 17 \end{pmatrix}$$

$$\xrightarrow[\substack{E4 \leftarrow E4-E3 \\ E5 \leftarrow E5+E3 \\ E6 \leftarrow E6-2E3}]{\substack{E1 \leftarrow E1-E3 \\ E2 \leftarrow E2+E3}} \begin{pmatrix} 1 & 0 & 0 & -1 & -8 \\ 0 & 1 & 0 & 1 & 18 \\ 0 & 0 & 1 & 1 & 20 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 2 & 23 \\ 0 & 0 & 0 & -2 & -23 \end{pmatrix}$$

$$E5 \leftarrow \tfrac{1}{2}E5 \longrightarrow \quad \begin{array}{ccccc} 1 & 0 & 0 & -1 & -8 \\ 0 & 1 & 0 & 1 & 18 \\ 0 & 0 & 1 & 1 & 20 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 1 & 23/2 \\ 0 & 0 & 0 & -2 & 23 \end{array}$$

$E1 \leftarrow E1 + E5$
$E2 \leftarrow E2 - E5$
$E3 \leftarrow E3 - E5$
$\longrightarrow$
$E6 \leftarrow E6 + 2E5$

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 0 & 13/2 \\ 0 & 0 & 1 & 0 & 17/2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 1 & 23/2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \hline a & b & c & d & = \text{cost.} \end{array}$$

$$\boxed{\begin{array}{l} a = 7/2 \\ b = 13/2 \\ c = 17/2 \\ d = 23/2 \end{array}}$$