Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 21 giugno 2017

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- *I candidati non parlino fra di loro!*
- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), scritto a mano dallo stesso candidato. È a disposizione anche una tavola per la distribuzione normale standardizzata.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

Le formule per le coordinate sferiche sono:

$$x = r\,\mathrm{sen}\,\theta\cos\phi \qquad y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\,\mathrm{sen}\,\theta\,dr\,d\theta\,d\phi$$

$\boxed{9}$

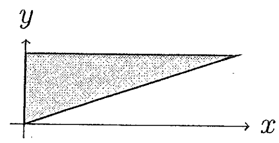
A. Calcolare

$$\int \frac{v}{v^4 - 16}\,dv$$

11

B. Siano $A, B > 0$ costanti e sia $\mathcal{T}$ il triangolo con vertici a $(0,0)$, $(0,A)$, e $\cancel{(A,B)}$. Usare le coordinate polari e calcolare
(B, A).

$$\iint_{\mathcal{T}} (x^2 + y^2)^{3/2} \, dA$$

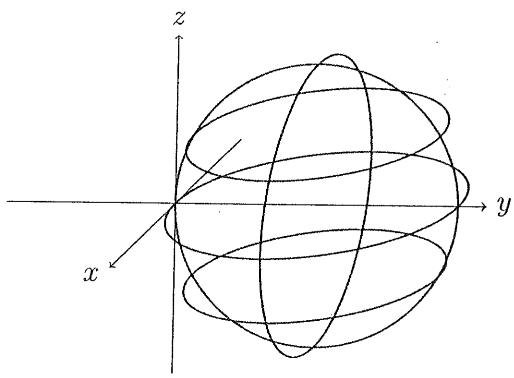


12

C. Sia $R > 0$ costante e sia $\mathcal{S}$ la sfera definita da:

$$x^2 + (y - R)^2 + z^2 \leq R^2$$

Usare le coordinate sferiche e calcolare la media di y^2 su $\mathcal{S}$.



8

D. Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio delle soluzioni del sistema:

$$\begin{array}{rrrrl} a & +2b & +4c & +5d & = 6 \\ 2a & +3b & +5c & +6d & = 7 \\ 3a & +4b & +7c & +8d & = 9 \\ 4a & +5b & +8c & +9d & = 10 \end{array}$$

(Indicazione: se il metodo è seguito senza errore e senza variazione, non saranno necessari denominatori.)

A $v^4 - 16 = (v^2 - 4)(v^2 + 4)$

$= (v-2)(v+2)(v^2+4)$

$$\frac{v}{v^4-16} = \frac{A}{v-2} + \frac{B}{v+2} + \frac{C}{v^2+4} + \frac{D(2v)}{v^2+4}$$

$$v = A(v+2)(v^2+4) + B(v-2)(v^2+4)$$

$$+ C(v-2)(v+2) + 2Dv(v-2)\cdot(v+2)$$

Mettiamo $v = 2$

$2 = A(4)(8) + 0 + 0 + 0$

$\rightarrow A = \frac{1}{16}$

Mettiamo $v = -2$

$-2 = 0 + B(-4)\cdot 8 + 0 + 0$

$\rightarrow B = \frac{1}{16}$

$$v = \frac{1}{16}(v^3 + 2v^2 + 4v + 8)$$
$$+ \frac{1}{16}(v^3 - 2v^2 + 4v - 8)$$
$$+ C(\quad v^2 \quad -4)$$
$$+ D(2v^3 \quad -8v \quad)$$

$$\rightsquigarrow \quad 0 = \frac{1}{16} + \frac{1}{16} + 2D$$
$$0 = \frac{1}{8} - \frac{1}{8} + C$$
$$1 = \frac{1}{4} + \frac{1}{4} - 8D$$
$$0 = \frac{1}{2} - \frac{1}{2} - 4C$$

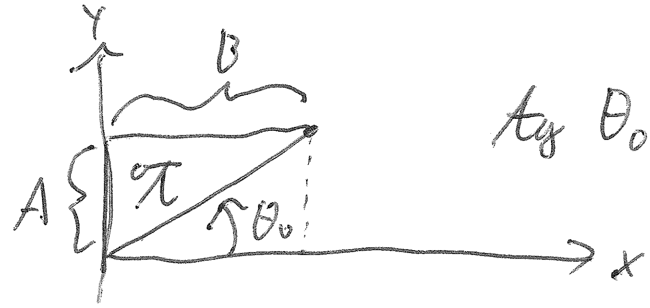
$$\longrightarrow \quad C = 0, \quad D = -\frac{1}{16}$$

$$\int \frac{v}{v^4 - 16}\, dv = \int \frac{1}{16}\frac{1}{v-2} + \frac{1}{16}\frac{1}{v+2}$$
$$- \frac{1}{16}\frac{2v}{v^2+4}\, dv$$

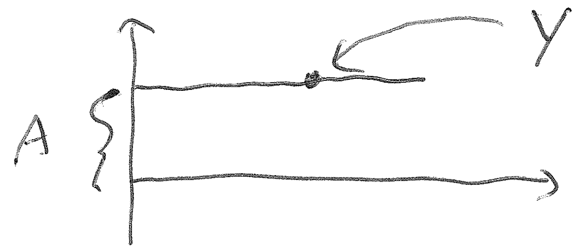
$$= \boxed{\frac{1}{16}\log(v-2) + \frac{1}{16}\log(v+2) - \frac{1}{16}\log(v^2+4)}$$

$$= \frac{1}{16}\log\left(\frac{v^2-4}{v^2+4}\right)$$

(B)



$tg\ \theta_0 = A/B$



$y = A \rightsquigarrow r\,\mathrm{sen}\,\theta = A$

$\rightsquigarrow r = A\,\mathrm{cosec}\,\theta$

$$\iint_{\mathcal{R}} (x^2+y^2)^{3/2}\, dA$$

$$= \iint_{\mathcal{R}} (r^2)^{3/2}\, r\, dr\, d\theta$$

$$= \int_{\theta_0}^{\pi/2} \left(\int_0^{A \csc \theta} r^3 \cdot r \, dr \right) d\theta$$

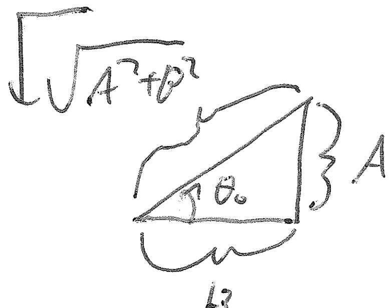
$$= \int_{\theta_0}^{\pi/2} \frac{1}{5} (A \csc \theta)^5 \, d\theta$$

$$= \frac{A^5}{5} \left(\left[-\frac{1}{4} \operatorname{ctg} \theta \csc^3 \theta \right]_{\theta_0}^{\pi/2} + \frac{3}{4} \int_{\theta_0}^{\pi/2} \csc^3 \theta \, d\theta \right)$$

$$= \frac{A^5}{5} \left(\frac{1}{4} \operatorname{ctg} \theta_0 \csc^3 \theta_0 + \frac{3}{4} \left(\left[-\frac{1}{2} \operatorname{ctg} \theta \csc \theta \right]_{\theta_0}^{\pi/2} + \frac{1}{2} \int_{\theta_0}^{\pi/2} \csc \theta \right) \right)$$

$$= \frac{A^5}{5} \left(\frac{1}{4} \operatorname{ctg} \theta_0 \csc^3 \theta_0 + \frac{3}{8} \operatorname{ctg} \theta_0 \csc \theta_0 + -\frac{3}{8} \Big[\log(\operatorname{ctg} \theta + \csc \theta) \Big]_{\theta_0}^{\pi/2} \right)$$

$$= A^5\left(\frac{1}{20}\operatorname{ctg}\theta_0\csc^3\theta_0 + \frac{3}{40}\operatorname{ctg}\theta_0\csc\theta_0 + \frac{3}{40}\log\left(\operatorname{ctg}\theta_0+\csc\theta_0\right)\right)$$



$$\rightsquigarrow \operatorname{ctg}\theta_0 = \frac{B}{A}$$

$$\csc\theta_0 = \frac{\sqrt{A^2+B^2}}{A}$$

$$= \boxed{\frac{AB(A^2+B^2)^{3/2}}{20} + \frac{3A^3B(A^2+B^2)^{1/2}}{40} + \frac{3A^5}{40}\log\left(\frac{B+\sqrt{A^2+B^2}}{A}\right)}$$

(c) $R^2 = x^2+(y-R)^2+z^2$

$$= x^2+y^2+z^2-2yR+R^2$$

$$\rightarrow 2yR \geq x^2+y^2+z^2$$

$$\rightarrow 2r \,\text{sen}\,\theta \,\text{sen}\,\varphi \geq r^2$$

$$\rightarrow 2R \,\text{sen}\,\theta \,\text{sen}\,\varphi \geq r$$

$$\iiint_S y^2 \, dV$$

$$= \int_0^{\pi} \left(\int_0^{\pi} \left(\int_0^{2R\,\text{sen}\,\theta\,\text{sen}\,\varphi} (r \,\text{sen}\,\theta \,\text{sen}\,\varphi)^2 \cdot r^2 \,\text{sen}\,\theta \, dr \right) d\varphi \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^{\pi} \left(\int_0^{2R\,\text{sen}\,\theta\,\text{sen}\,\varphi} r^4 \,\text{sen}^3\theta \,\text{sen}^2\varphi \, dr \right) d\varphi \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^{\pi} \frac{(2R\,\text{sen}\,\theta\,\text{sen}\,\varphi)^5}{5} \text{sen}^3\theta \,\text{sen}^2\varphi \right) d\varphi \Big) d\theta$$

$$= \frac{32}{5} R^5 \int_0^{\pi} \left(\int_0^{\pi} \text{sen}^8\theta \,\text{sen}^7\varphi \right) d\varphi \Big) d\theta$$

[$\displaystyle\int_0^{\pi} \operatorname{sen}^8\theta \, d\theta = -\frac{1}{8}\cos\theta \operatorname{sen}^7\theta \Big]_0^{\pi} + \frac{7}{8}\int_0^{\pi} \operatorname{sen}^6\theta \, d\theta$

$\displaystyle = \frac{7}{8}\left(-\frac{1}{6}\cos\theta \operatorname{sen}^5\theta \Big]_0^{\pi} + \frac{5}{6}\int_0^{\pi} \operatorname{sen}^4\theta \, d\theta \right)$

$\displaystyle = \frac{7}{8}\cdot\frac{5}{6}\left(-\frac{1}{4}\cos\theta \operatorname{sen}^3\theta \Big]_0^{\pi} + \frac{3}{4}\int_0^{\pi} \operatorname{sen}^2\theta \, d\theta \right)$

$\displaystyle = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\left(-\frac{1}{2}\cos\theta \operatorname{sen}\theta \Big]_0^{\pi} + \frac{1}{2}\int_0^{\pi} 1 \, d\theta \right)$

$\displaystyle = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi$

]

$$\left[\int_0^{\pi} \operatorname{sen}^7 \varphi \, d\varphi = \frac{6}{7} \int_0^{\pi} \operatorname{sen}^5 \varphi \, d\varphi \right.$$

$$= \frac{6}{7} \cdot \frac{4}{5} \cdot \int_0^{\pi} \operatorname{sen}^3 \varphi \, d\varphi$$

$$= \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi} \operatorname{sen} \varphi \, d\varphi$$

$$\left. = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 2 \right]$$

$$\iiint_{S} y^2 \, dV$$

$$= \frac{32 R^5}{5} \int_0^{\pi} \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 2 \operatorname{sen}^8 \theta \, d\theta$$

$$= \frac{32 R^5}{5} \cdot \frac{\cancel{6}}{\cancel{7}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot 2 \cdot \frac{\cancel{7}}{8} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{1}{\cancel{2}} \pi$$

$$= \frac{8 \pi R^5}{5}$$

p. 9/10

$$\overline{Y^2} = \frac{\iiint_{\mathcal{S}} Y^2 \, dV}{\iiint_{\mathcal{S}} 1 \, dV}$$

$$= \frac{8\pi R^5/5}{4\pi R^3/3} = \frac{6\pi R^2}{5}$$

D

a	b	c	d	= cost.
1	2	4	5	6
2	3	5	6	7
3	4	7	8	9
4	5	8	9	10

$$\begin{pmatrix} E2 \leftarrow E2 - 2E1 \\ E3 \leftarrow E3 - 3E1 \\ E4 \leftarrow E4 - 4E1 \end{pmatrix}$$

$$\begin{array}{ccccc} 1 & 2 & 4 & 5 & 6 \\ 0 & -1 & -3 & -4 & -5 \\ 0 & -2 & -5 & -7 & -9 \\ 0 & -3 & -8 & -11 & -14 \end{array}$$

$E2 \leftarrow -E2$

$$\begin{array}{ccccc} 1 & 2 & 4 & 5 & 6 \\ 0 & 1 & 3 & 4 & 5 \\ 0 & -2 & -5 & -7 & -9 \\ 0 & -3 & -8 & -11 & -14 \end{array}$$

p.10/10

$$\xrightarrow[\substack{E3 \leftarrow E3 + 2E2 \\ E4 \leftarrow E4 + 3E2}]{E1 \leftarrow E1 - 2E2} \begin{matrix} 1 & 0 & -2 & -3 & -4 \\ 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{matrix}$$

$$\xrightarrow[E4 \leftarrow E4 - E3]{\substack{E1 \leftarrow E1 + 2E3 \\ E2 \leftarrow E2 - 3E3}} \begin{matrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{matrix}$$

$$a \quad -d = -2$$
$$b \quad +d = 2$$
$$c \quad +d = 1$$

$$\begin{cases} a = -2 + d \\ b = 2 - d \\ c = 1 - d \\ d = d \end{cases}$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$