DA SCANNERIZZARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 11 settembre 2024

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z \qquad\qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Le formule per le coordinate sferiche sono:

$$x = r\,\mathrm{sen}\,\theta\cos\phi \qquad y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\,\mathrm{sen}\,\theta\,dr\,d\theta\,d\phi$$

A. Usare la sostituzione $u = e^t$ e calcolare

$$\int \frac{1}{(e^t + e^{-t})^2}\,dt$$

B. Siano $0 \le A \le R$ due parametri fissi. Si consideri la forma determinata dalle disuguaglianze:

$$x^2 + y^2 \le A^2 \qquad\qquad x^2 + y^2 + z^2 \le R^2$$

Fare uno schizzo di questa forma, e usare le coordinate cilindriche per calcolare il suo volume.

C. Sia $\mathcal{O}$ l'ottava parte della sfera determinata dalle disuguaglianze

$$x^2 + y^2 + z^2 \leq R^2 \qquad\qquad x, y, z \geq 0$$

Usare le coordinate sferiche, calcolare $\iiint_{\mathcal{O}} z^4 \, dV$ e $\iiint_{\mathcal{O}} x^4 \, dV$, e verificare che siano uguali.

D. Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio delle soluzioni del sistema:

$$\begin{aligned}
p + q + r &= 0 \\
p + r + s &= 0 \\
p + s + t &= 0 \\
p + t + q &= 0 \\
q + r + u &= 0 \\
r + s + u &= 0 \\
s + t + u &= 0 \\
t + q + u &= 0
\end{aligned}$$

$\boxed{A}$ $u = e^t, \quad du = e^t\,dt$

$$\int \frac{1}{(e^t+e^{-t})^2}\,dt = \int \frac{1}{(e^t+e^{-t})^2\,e^t}\,e^t\,dt$$

$$= \int \frac{1}{(u+u^{-1})^2\,u}\,du = \int \frac{u}{(u+u^{-1})^2\,u^2}\,du$$

$$= \int \frac{u}{(u^2+1)^2}\,du = -\frac{1}{2}\,\frac{1}{u^2+1}$$

(È già una delle integrande semplici: non è il caso di decomporla.)

$$= -\frac{1}{2}\,\frac{1}{(e^t)^2+1} = -\frac{1}{2}\,\frac{1}{e^{2t}+1}$$

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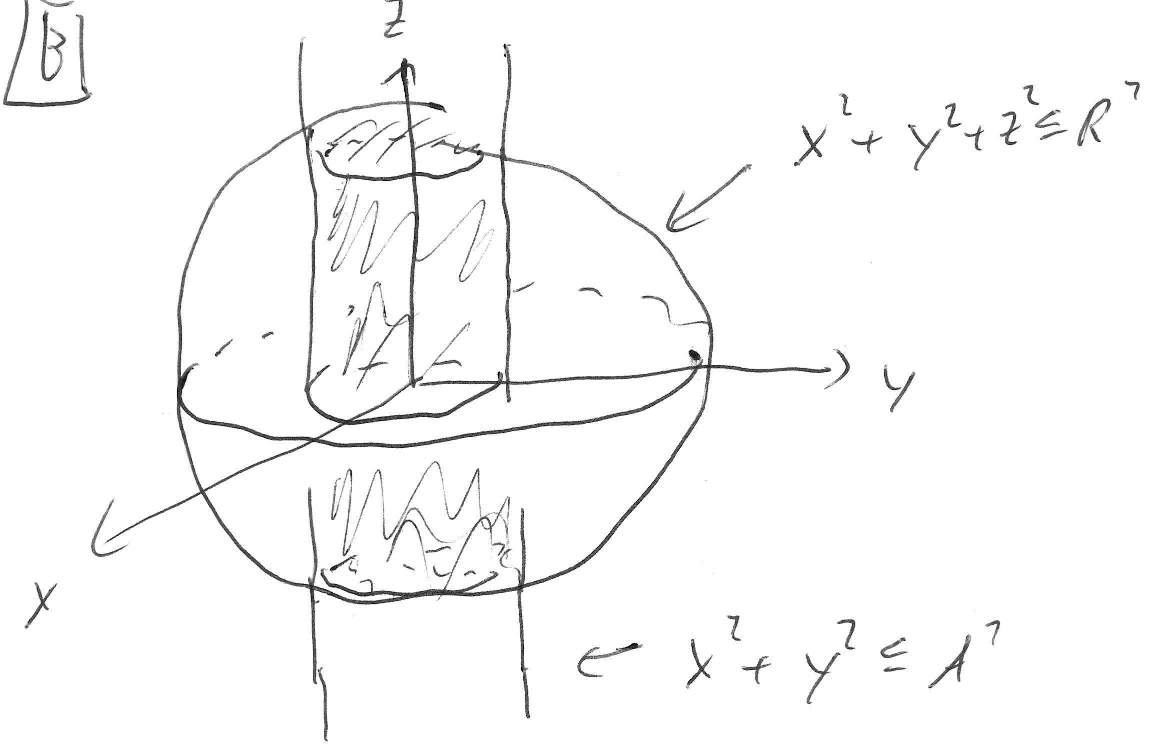
B



La forma è un cilindro con calotte sopra e sotto.

$$x^2+y^2 \le A^2 \leftrightarrow (r\cos\theta)^2+(r\,\mathrm{sen}\,\theta)^2 \le A^2$$
$$\leftrightarrow r^2 \le A^2 \leftrightarrow r \le A$$

$$x^2+y^2+z^2 \le R^2 \leftrightarrow r^2+z^2 \le R^2$$
$$\leftrightarrow z^2 \le R^2-r^2$$
$$\leftrightarrow |z| \le \sqrt{R^2-r^2}$$

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$$\iiint_{\text{boule}} 1 \, dV$$

$$= \int_{\theta=-\pi}^{\pi} \left(\int_{r=0}^{A} \left(\int_{z=-\sqrt{R^2-r^2}}^{+\sqrt{R^2-r^2}} 1 \cdot r \, dz \right) dr \right) d\theta$$

$$= 2\pi \int_{r=0}^{A} 2r\sqrt{R^2-r^2} \, dr$$

$$= 4\pi \int_{r=0}^{A} r\,(R^2-r^2)^{1/2} \, dr$$

$$= 4\pi \cdot \frac{-1}{2} \cdot \frac{2}{3} (R^2-r^2)^{3/2} \Big]_{r=0}^{A}$$

$$= -\frac{4\pi}{3} \left((R^2-A^2)^{3/2} - (R^2)^{3/2} \right)$$

$$= \frac{4\pi}{3} \left(R^3 - (R^2-A^2)^{3/2} \right)$$

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$\boxed{C}$ $x^2+y^2+z^2 \le R^2 \leftrightarrow \begin{bmatrix} r^2 \le R^2 \\ 0 \le r \end{bmatrix}$

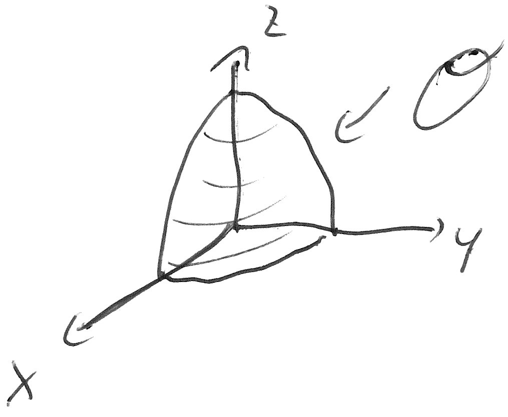
$\leftrightarrow 0 \le r \le R$

$z \ge 0 \leftrightarrow r\cos\theta \ge 0 \leftrightarrow \begin{bmatrix} \cos\theta \ge 0 \\ 0 \le \theta \le \pi \end{bmatrix}$

$\leftrightarrow 0 \le \theta \le \pi/2$

$\begin{bmatrix} x \ge 0 \\ y \ge 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} r \operatorname{sen}\theta \cos\varphi \ge 0 \\ r \operatorname{sen}\theta \operatorname{sen}\varphi \ge 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} \cos\varphi \ge 0 \\ \operatorname{sen}\varphi \ge 0 \\ -\pi \le \varphi \le \pi \end{bmatrix}$

$\leftrightarrow 0 \le \varphi \le \pi/2$



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$$\iiint_{\theta} z^4 \, d\nabla = \int_{\varphi=0}^{\pi/2} \left(\int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{R} \right. \right.$$

$$\left. \left. \cdot (r \cos\theta)^4 \, r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \frac{\pi}{2} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta \cos^4\theta \left(\int_{r=0}^{R} r^6 \, dr \right) d\theta$$

$$= \frac{\pi}{2} \int_{\theta=0}^{\pi/2} \operatorname{sen}\theta \cos^4\theta \, \frac{R^7}{7} \, d\theta$$

$$= \frac{\pi}{2} \cdot \frac{R^7}{7} \cdot \frac{1}{5} (-\cos\theta)^5 \Big]_{\theta=0}^{\pi/2}$$

$$= \frac{\pi}{2} \cdot \frac{R^7}{7} \cdot \frac{1}{5} = \frac{\pi R^7}{70}$$

$$\iiint_{\theta} x^4 \, d\nabla$$

$$= \int_{\varphi=0}^{\pi/2} \left(\int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{R} (r \operatorname{sen}\theta \cos\varphi)^4 \, r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \cos^4 \varphi \left(\int_{\theta=0}^{\pi/2} \operatorname{sen}^5 \theta \left(\int_{r=0}^{R} r^6 \, dr \right) d\theta \right) d\varphi$$

$$\Big[\quad \int_{\varphi=0}^{\pi/2} \cos^4 \varphi \, d\varphi = \frac{1}{4} \cancelto{0}{\operatorname{sen} \varphi \cos^3 \varphi} \Big]_{\varphi=0}^{\pi/2}$$

$$+ \frac{3}{4} \int_{\varphi=0}^{\pi/2} \cos^2 \varphi \, d\varphi$$

$$= \frac{3}{4} \left(\frac{1}{2} \cancelto{0}{\operatorname{sen} \varphi \cos \varphi} \Big]_{\varphi=0}^{\pi/2} + \frac{1}{2} \int_{\varphi=0}^{\pi/2} 1 \, d\varphi \right)$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16}$$

$$\int_{\theta=0}^{\pi/2} \operatorname{sen}^5 \theta \, d\theta = \frac{4}{5} \int_{\theta=0}^{\pi/2} \operatorname{sen}^3 \theta \, d\theta$$

$$= \frac{4}{5} \cdot \frac{2}{3} \int_{\theta=0}^{\pi/2} \operatorname{sen} \theta \, d\theta = \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$= \frac{8}{15}$$

$$\Big]$$

$$\iiint_{Q} x^4 \, dV$$

$$= \int_{\varphi=0}^{\pi/2} \cos^4\varphi \left(\int_{\theta=0}^{\pi/2} \operatorname{sen}^9\theta \cdot \frac{R^7}{7} \, d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \cos^4\varphi \cdot \frac{8}{15} \cdot \frac{R^7}{7} \, d\varphi$$

$$= \frac{3\pi}{16} \cdot \frac{8}{15} \cdot \frac{R^7}{7} = \frac{\pi R^7}{70}$$

$\boxed{D}$ Questo sistema è particolarmente facile ad affrontare, anche senza Gauss-Jordan, ma ecco l'approccio meccanico:

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p	q	r	s	t	u = 0
1	1	1	0	0	0
1	0	1	1	0	0
1	0	0	1	1	0
1	0	0	0	1	0
0	1	1	0	0	1
0	0	1	1	0	1
0	0	0	1	1	1
0	1	0	0	1	1

$E2 \leftarrow E2 - E1$
$E3 \leftarrow E3 - E1$
$\longrightarrow$
$E4 \leftarrow E4 - E1$

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

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$E2 \leftarrow -E2$

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{matrix}$$

$E1 \leftarrow E1 - E2$
$E3 \leftarrow E3 + E2$
$E5 \leftarrow E5 - E2$
$E8 \leftarrow E8 - E2$

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}$$

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$$E_3 \xrightarrow{\quad} -E_3 \qquad \begin{matrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}$$

$E_1 \leftarrow E_1 - E_3$

$E_4 \leftarrow E_4 + E_3$

$\longrightarrow$

$E_5 \leftarrow E_5 - E_3$

$E_6 \leftarrow E_6 - E_3$

$$\begin{matrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix} \quad \begin{matrix} \\ \\ \\ \rightarrow [0=0] \\ \\ \\ \\ \\ \end{matrix}$$

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$E1 \leftarrow E1 - E4$

$E2 \leftarrow E2 + E4$

$\longrightarrow$

$E5 \leftarrow E5 - E4$

$E6 \leftarrow E6 - E4$

$E7 \leftarrow E7 - E4$

p	q	r	s	t	u	=0
1	0	0	0	0	-1	
0	1	0	0	1	1	
0	0	1	0	-1	0	
0	0	0	1	1	1	
~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	
~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	
~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	

$$\begin{cases} p - u = 0 \\ q + t + u = 0 \\ r - t = 0 \\ s + t + u = 0 \end{cases} \rightarrow \begin{cases} p = u \\ q = -t-u \\ r = t \\ s = -t-u \end{cases}$$

$$\rightarrow \boxed{\begin{pmatrix} p \\ q \\ r \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} u \\ -t-u \\ t \\ -t-u \\ t \\ u \end{pmatrix}}$$

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Controllo:

$$p+q+r = u+(-t-u)+t=0$$
$$p+r+s = u+t+(-t-u)=0$$
$$p+s+t = u+(-t-u)+t=0$$
$$p+t+q = u+t+(-t-u)=0$$
$$q+r+u = (-t-u)+t+u=0$$
$$r+s+u = t+(-t-u)+u=0$$
$$s+t+u = (-t-u)+t+u=0$$
$$t+q+u = t+(-t-u)+u=0$$