DA SCANNERIZZARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 11 febbraio 2025

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad z = z \qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Le formule per le coordinate sferiche sono:

$$x = r \operatorname{sen}\theta\cos\phi \qquad y = r\operatorname{sen}\theta\operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2 \operatorname{sen}\theta\,dr\,d\theta\,d\phi$$

A. Calcolare

$$\int \frac{z^3}{(z-2)(z-3)(z-5)(z-7)}\,dz$$

B. Usare le coordinate *cilindriche* e calcolare la media di z sul tetraedro $\mathcal{T}$ definito da $0 \le z \le y \le x \le 1$.

C. Siano R e A due parametri fissi con $0 < A < R$. Sia $\mathcal{G}$ la forma definita da:

$$z \geq 0 \qquad x^2 + y^2 + z^2 \leq R^2 \qquad x^2 + y^2 \leq \left(\frac{R^2}{A^2} - 1\right) z^2$$

Prima fare uno schizzo della forma $\mathcal{G}$, con attenzione particolare alla circonferenza dove $x^2+y^2+z^2 = R^2$ e $x^2+y^2 = \left(\frac{R^2}{A^2} - 1\right) z^2$. Poi usare le coordinate *sferiche* e calcolare il volume di $\mathcal{G}$.

D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema:

$$\begin{array}{rrrl} x & +4y & +9z & = 0 \\ 4x & +9y & +16z & = 2 \\ 9x & +16y & +25z & = 0 \end{array}$$

(Indicazione: nel corso dei calcoli viene fuori il denominatore 7 ma questo sparisce nella risposta finale.)

1/10

$\boxed{A}$ $$\frac{z^3}{(z-2)(z-3)(z-5)(z-7)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z-5} + \frac{D}{z-7}$$

$$z^3 = A(z-3)(z-5)(z-7) + B(z-2)(z-5)(z-7) + C(z-2)(z-3)(z-7) + D(z-2)(z-3)(z-5)$$

$z=2 \to 8 = A(2-3)(2-5)(2-7) + 0 + 0 + 0$

$\hookrightarrow 8 = A(-1)(-3)(-5) \to A = -8/15$

$z=3 \to 27 = 0 + B(3-2)(3-5)(3-7) + 0 + 0$

$\hookrightarrow 27 = B(1)(-2)(-4) \to B = +27/8$

2/10

$z=5 \rightsquigarrow 125 = 0+0+C(5-2)(5-3)(5-7)+0$

$\hookrightarrow 125 = C(3)(2)(-2) \rightarrow C = -125/12$

$z=7 \rightsquigarrow 343 = 0+0+0+D(7-2)(7-3)(7-5)$

$\hookrightarrow 343 = D(5)(4)(2) \rightarrow D = +343/40$

$$\int -\frac{8}{15}\frac{1}{z-2} + \frac{27}{8}\frac{1}{z-3} - \frac{125}{12}\frac{1}{z-5} + \frac{343}{40}\frac{1}{z-7}\, dz$$

$$= -\frac{8}{15}\log(z-2) + \frac{27}{8}\log(z-3) - \frac{125}{12}\log(z-5) + \frac{343}{40}\log(z-7)$$

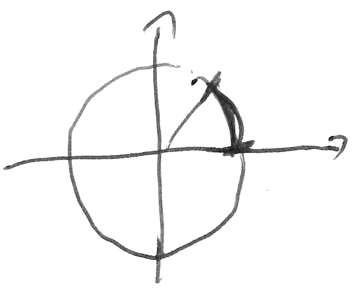
$\boxed{3}$ $x \leq 1 \Leftrightarrow r\cos\theta \leq 1 \Leftrightarrow r \leq \frac{1}{\cos\theta} = \sec\theta$

$r \geq 0$ sempre

$0 \leq y \leq x \Leftrightarrow 0 \leq r\,\mathrm{sen}\,\theta \leq r\cos\theta$

$\Leftrightarrow 0 \leq \mathrm{sen}\,\theta \leq \cos\theta \Leftrightarrow 0 \leq \theta \leq \frac{\pi}{4}$

3/10



$0 \le z \le y$

$\hookrightarrow 0 \le z \le r \,\text{sen}\,\theta$

$$\iiint_{\mathcal{R}} 1 \, dV$$

$$= \int_{\theta=0}^{\pi/4} \left(\int_{r=0}^{\sec\theta} \left(\int_{z=0}^{r\,\text{sen}\,\theta} 1 \cdot r \, dz \right) dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi/4} \left(\int_{r=0}^{\sec\theta} r^2 \,\text{sen}\,\theta \, dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{1}{3} \sec^3\theta \,\text{sen}\,\theta \, d\theta = \frac{1}{3} \int_{\theta=0}^{\pi/4} \frac{\text{sen}\,\theta}{\cos^3\theta} \, d\theta$$

$$= \frac{1}{3} \cdot \frac{1}{2} (\cos\theta)^{-2} \Big]_{\theta=0}^{\pi/4} = \frac{1}{6} \left(\frac{1}{(\sqrt{2}/2)^2} - \frac{1}{1^2} \right)$$

$$= \frac{1}{6}(2-1) = 1/6$$

4/10

$$\iiint_{\mathcal{R}} z \, dV$$

$$= \int_{\theta=0}^{\pi/4} \left(\int_{r=0}^{\sec\theta} \left(\int_{z=0}^{r\,\mathrm{sen}\,\theta} z \cdot r \, dz \right) dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi/4} \left(\int_{r=0}^{\sec\theta} \frac{1}{2} (r\,\mathrm{sen}\,\theta)^2 \, r\,dr \right) d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/4} \mathrm{sen}^2\theta \left(\int_{r=0}^{\sec\theta} r^3 \, dr \right) d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/4} \mathrm{sen}^2\theta \cdot \frac{1}{4} \sec^4\theta \, d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{4} \int_{\theta=0}^{\pi/4} \frac{\mathrm{sen}^2\theta}{\cos^4\theta} \, d\theta$$

$$= \frac{1}{8} \int_{\theta=0}^{\pi/4} \frac{1-\cos^2\theta}{\cos^4\theta} \, d\theta = \frac{1}{8} \int_{\theta=0}^{\pi/4} \sec^4\theta - \sec^2\theta \, d\theta$$

5/10

$$\int_{\theta=0}^{\pi/4} \sec^2\theta \, d\theta = \operatorname{tg}\theta \Big]_{\theta=0}^{\pi/4} = 1 - 0 = 1$$

$$\int_{\theta=0}^{\pi/4} \sec^4\theta \, d\theta = \frac{1}{3}\operatorname{tg}\theta \sec^2\theta \Big]_{\theta=0}^{\pi/4} + \frac{2}{3}\int_{\theta=0}^{\pi/4} \sec^2\theta \, d\theta$$

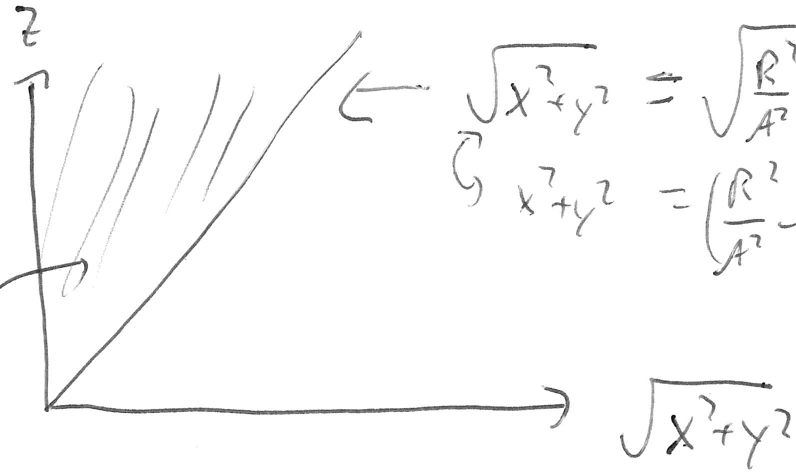
$$= \frac{1}{3}\left(1\cdot(\sqrt{2})^2 - 0\right) + \frac{2}{3} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$\iiint_{\mathcal{R}} z \, dV = \frac{1}{8}\left(\frac{4}{3} - 1\right) = \frac{1}{8}\cdot\frac{1}{3} = \frac{1}{24}$$

$$\bar{z} = \frac{\iiint_{\mathcal{R}} z \, dV}{\iiint_{\mathcal{R}} 1 \, dV} = \frac{1/24}{1/6} = \frac{1}{4}$$

6/10

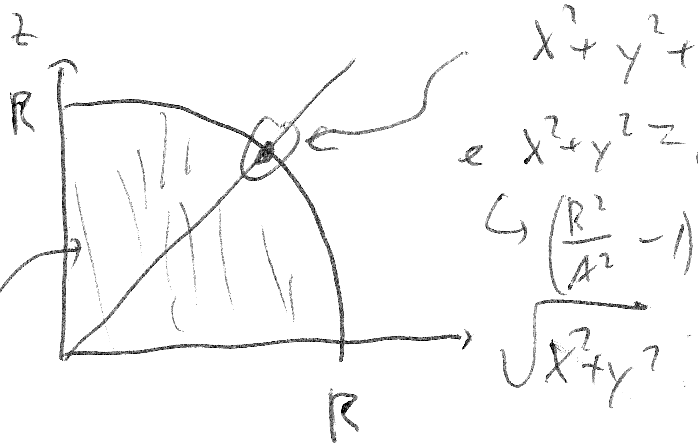
$\boxed{C}$



$\sqrt{x^2+y^2} = \sqrt{\frac{R^2}{A^2}-1}\cdot z$

$\hookrightarrow x^2+y^2 = \left(\frac{R^2}{A^2}-1\right)\cdot z^2$

$x^2+y^2 \le \left(\frac{R^2}{A^2}-1\right)\cdot z^2$



$x^2+y^2+z^2=R^2$

$\& \; x^2+y^2=\left(\frac{R^2}{A^2}-1\right)\cdot z^2$

$\hookrightarrow \left(\frac{R^2}{A^2}-1\right)\cdot z^2+z^2=R^2$

$\Updownarrow$

$\frac{R^2}{A^2} z^2 = R^2$

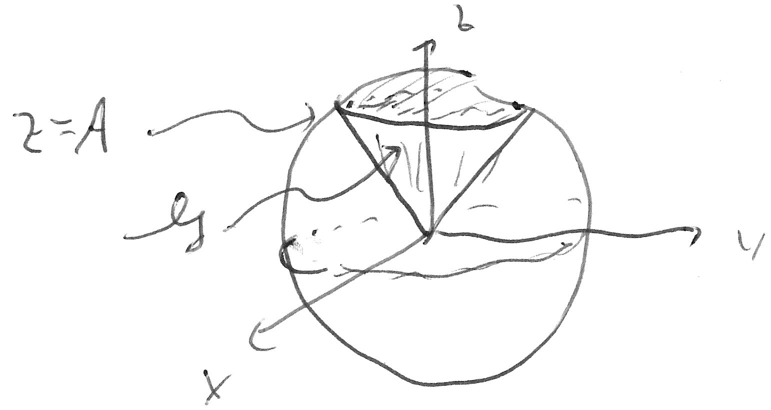
$\Updownarrow$

$z^2=A^2$

$\Updownarrow$

$z=A$

$x^2+y^2+z^2 \le R^2$



7/16

$x^2+y^2+z^2 \leq R^2 \rightsquigarrow r^2 \leq R^2 \rightsquigarrow r \leq R$

$r \geq 0$ sempre

$z \geq 0 \rightsquigarrow r\cos\theta \geq 0 \rightsquigarrow \cos\theta \geq 0$

$\rightarrow \theta \leq \frac{\pi}{2}$

$x^2+y^2 \leq \left(\frac{R^2}{A^2}-1\right) z^2$

$\rightsquigarrow r^2 \operatorname{sen}^2\theta \leq \left(\frac{R^2}{A^2}-1\right) r^2 \cos^2\theta$

$\rightsquigarrow \left(\frac{\operatorname{sen}\theta}{\cos\theta}\right)^2 \leq \left(\frac{R^2}{A^2}-1\right)$

$\rightarrow \operatorname{tg}\theta \leq \sqrt{\frac{R^2}{A^2}-1}$

$\rightarrow \theta \leq \theta_0 = \operatorname{arctg}\left(\sqrt{\frac{R^2}{A^2}-1}\right)$

8/16

$$vol(\mathcal{G}) = \iiint_{\mathcal{G}} 1 \, dV$$

$$= \int_{\varphi=-\pi}^{\pi} \left(\int_{\theta=0}^{\theta_0} \left(\int_{\rho=0}^{R} 1 \cdot \rho^2 \operatorname{sen}\theta \, d\rho \right) d\theta \right) d\varphi$$

$$= 2\pi \left(\int_{\theta=0}^{\theta_0} \frac{R^3}{3} \operatorname{sen}\theta \, d\theta \right)$$

$$= -\frac{2\pi}{3} R^3 \cos\theta \Big]_{\theta=0}^{\theta_0}$$

$$\operatorname{tg}\theta_0 = \sqrt{\frac{R^2}{A^2} - 1} \to \sec^2\theta_0 = 1 + \operatorname{tg}^2\theta_0$$

$$= 1 + \left(\frac{R^2}{A^2} - 1\right) = \frac{R^2}{A^2}$$

$$\sec\theta_0 = \frac{R}{A} \to \cos\theta_0 = \frac{A}{R}$$

$$vol(\mathcal{G}) = \frac{2\pi}{3} R^3 \left(-\frac{A}{R} + 1\right) = \frac{2\pi}{3}(R^3 - AR^2)$$

9/10

[D]

x	y	z	=
1	4	9	0
4	9	16	2
9	16	25	0

$\xrightarrow[E3 \leftarrow E3 - 9E1]{E2 \leftarrow E2 - 4E1}$

$$\begin{matrix} 1 & 4 & 9 & 0 \\ 0 & -7 & -20 & 2 \\ 0 & -20 & -56 & 0 \end{matrix}$$

$\xrightarrow{E2 \leftarrow -E2/7}$

$$\begin{matrix} 1 & 4 & 9 & 0 \\ 0 & 1 & 20/7 & -2/7 \\ 0 & -20 & -56 & 0 \end{matrix}$$

$\xrightarrow[E3 \leftarrow E3 + 20 E2]{E1 \leftarrow E1 - 4E2}$

$$\begin{matrix} 1 & 0 & -17/7 & 8/7 \\ 0 & 1 & 20/7 & -2/7 \\ 0 & 0 & 8/7 & -40/7 \end{matrix}$$

$\xrightarrow{E3 \leftarrow \frac{7}{8} E3}$

$$\begin{matrix} 1 & 0 & -17/7 & 8/7 \\ 0 & 1 & 20/7 & -2/7 \\ 0 & 0 & 1 & -5 \end{matrix}$$

10/10

$E1 \xrightarrow{} E1 + \frac{17}{7} E3$

$E2 \leftarrow E2 - \frac{20}{7} E3$

[$\frac{8}{7} + \frac{17}{7}(-5) = \frac{8-85}{7} = \frac{-77}{7} = -11$

$-\frac{2}{7} - \frac{20}{7}(-5) = \frac{-2+100}{7} = \frac{98}{7} = 14$]

x	y	z	$=$
1	0	0	-11
0	1	0	14
0	0	1	-5

$x = -11$
$y = 14$
$z = -5$

<u>Controllo</u>: $x + 4y + 9z = -11 + 4(14) + 9(-5)$
$= -11 + 56 - 45 = 0$ ✓

$4x + 9y + 16z = 4(-11) + 9(14) + 16(-5)$
$= -44 + 126 - 80 = 2$ ✓

$9x + 16y + 25z = 9(-11) + 16(14) + 25(-5)$
$= -99 + 224 - 125 = 0$ ✓