DA SCANNERIZZARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 30 aprile 2025

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.

Le formule per le coordinate sferiche sono:

$$x = r \operatorname{sen}\theta \cos\phi \qquad y = r \operatorname{sen}\theta \operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\phi$$

A. Calcolare

$$\int \frac{1}{w^4 + w}\, dw$$

B. Usare le coordinate rettangolari (x, y, e z) e calcolare la media di z sulla forma definita dalle disuguaglianze:

$$x, y, z \geq 0 \qquad\qquad x + y + z \leq 1$$

C. Sia $R > 0$ un parametro fisso e sia $\mathcal{S}_1$ la sfera con raggio R e centro a $\left(\begin{smallmatrix} 0 \\ R \\ 0 \end{smallmatrix}\right)$. Prima fare uno schizzo di $\mathcal{S}_1$. Poi calcolare la media di y su $\mathcal{S}_1$.

D. Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio delle soluzioni del sistema:

$$\begin{array}{rrrrl} v & +2x & +3y & +4z & = 5 \\ 5v & +6x & +7y & +8z & = 4 \\ 9v & +10x & +11y & +12z & = 3 \\ 13v & +14x & +15y & +16z & = 2 \\ 17v & +18x & +19y & +20z & = 1 \\ 21v & +22x & +23y & +24z & = 0 \end{array}$$

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$\boxed{A}$ $$\frac{1}{w^4+w} = \frac{1}{w(w^3+1)} = \frac{1}{w(w+1)(w^2-w+1)}$$

$$= \frac{A}{w} + \frac{B}{w+1} + \frac{C}{w^2-w+1} + \frac{D(2w-1)}{w^2-w+1}$$

$$\rightarrow 1 = A(w+1)(w^2-w+1) + Bw(w^2-w+1)$$
$$+ Cw(w+1) + D(2w-1)w(w+1)$$

$w=0 \rightarrow 1 = A(0+1)(0^2-0+1)+0+0+0$

$\rightarrow 1 = A \cdot 1 \rightarrow A=1$

$w=-1 \quad 1 = 0 + B(-1)((-1)^2-(-1)+1)+0+0$

$\rightarrow 1 = B(-3) \rightarrow B=-1/3$

$$1 = (w+1)(w^2-w+1) - \frac{1}{3}w(w^2-w+1)$$
$$+ Cw(w+1) + D(2w-1)w(w+1)$$

$$\begin{aligned} = &(w^3 + 1) \\ &(-\tfrac{1}{3}w^3 + \tfrac{1}{3}w^2 - \tfrac{1}{3}w \\ &(\quad + Cw^2 + Cw \quad) \\ &(2Dw^3 + Dw^2 - Dw \quad) \end{aligned} \rightarrow \left\{ \begin{aligned} 0 &= 1 - \tfrac{1}{3} + 2D \\ 0 &= \tfrac{1}{3} + C + D \\ 0 &= -\tfrac{1}{3} + C - D \\ 1 &= 1 \end{aligned} \right.$$

$\rightarrow 2D = -\frac{2}{3} \rightarrow D = -\frac{1}{3}$

$0 = \frac{1}{3} + C - \frac{1}{3} \rightarrow C=0$

$0 = -\frac{1}{3} + C + \frac{1}{3} \rightarrow C=0$

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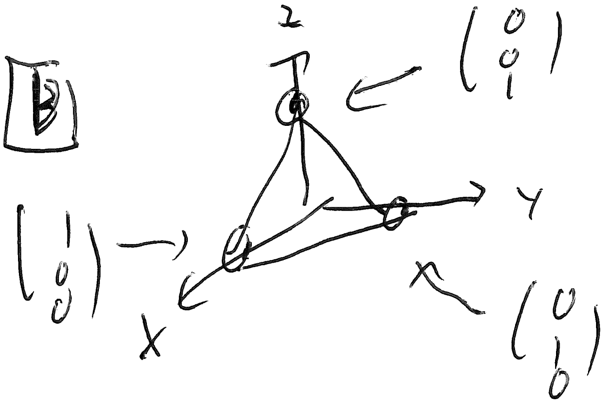
$$\int 1 - \frac{1}{w} - \frac{1}{3}\frac{1}{w+1} - \frac{1}{3}\frac{(2w-1)}{w^2-w+1}\,dw$$

$$= \log(w) - \frac{1}{3}\log(w+1) - \frac{1}{3}\log(w^2-w+1)$$

$\boxed{B}$



$$x+y+z \leq 1 \rightsquigarrow z \leq 1-x-y \rightsquigarrow 0 \leq z \leq 1-x-y$$

$$1-x-y \geq 0 \rightsquigarrow 0 \leq y \leq 1-x$$

$$1-x \geq 0 \rightarrow 0 \leq x \leq 1$$

$$\iiint_{\substack{x,y,z \geq 0 \\ x+y+z \leq 1}} 1 \, dV = \int_{x=0}^{1} \left(\int_{y=0}^{1-x} \left(\int_{z=0}^{1-x-y} 1 \, dz \right) dy \right) dx$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{1-x} 1-x-y \; dy \right) dx$$

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$$= \int_{x=0}^{1} \left[(1-x)y - \frac{y^2}{2} \right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} (1-x)^2 - \frac{(1-x)^2}{2} dx = \frac{1}{2} \int_{x=0}^{1} (1-x)^2 dx$$

$1-x=u$
$u=1-x$
$du=-dx$
$x=1 \leftrightarrow u=0$
$x=0 \leftrightarrow u=1$

$$= \frac{1}{2} \int_{u=1}^{0} u^2 \, (-du)$$

$$= \frac{1}{2} \int_{u=0}^{1} u^2 \, du = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$

$$\int_{x=0}^{1} \left(\int_{y=0}^{1-x} \left(\int_{z=0}^{1-x-y} z \, dz \right) dy \right) dx$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{1-x} \frac{1}{2} (1-x-y)^2 \, dy \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left(\int_{y=0}^{1-x} \left((1-x)^2 - 2y(1-x) + y^2 \right) dy \right) dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \left[(1-x)^2 y - (1-x)y^2 + \frac{y^3}{3}\right]_{y=0}^{1-x} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} (1-x)^3 - (1-x)^3 + \frac{1}{3}(1-x)^3 \, dx$$

$$= \frac{1}{2}\cdot\frac{1}{3}\int_{x=0}^{1} (1-x)^3 \, dx$$

$$= \frac{1}{2}\cdot\frac{1}{3}\int_{u=0}^{1} u^3 \, du = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}$$

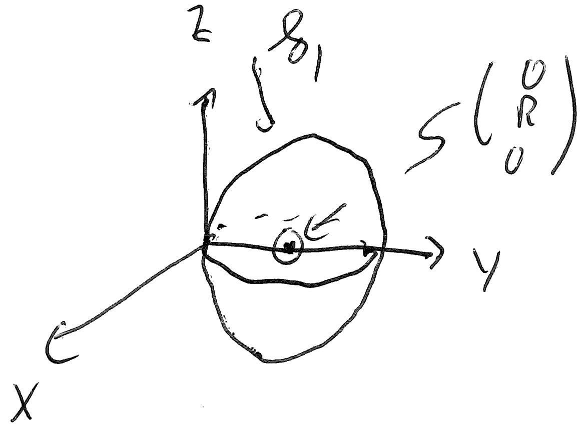
$$= \frac{1}{24}$$

$$\bar{z} = \frac{\displaystyle\iiint_{\substack{x,y,z\geq 0\\ x+y+z\leq 1}} z \, dV}{\displaystyle\iiint_{\substack{x,y,z\geq 0\\ x+y+z\leq 1}} 1 \, dV} = \frac{\frac{1}{24}}{\frac{1}{6}} = \frac{6}{24}$$

$$= \frac{1}{4}$$

(5/12)

$\boxed{C}$



$$\text{dist}\left(\begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}0\\R\\0\end{pmatrix}\right) \leq R$$

$$\hookrightarrow \sqrt{(x-0)^2+(y-R)^2+(z-0)^2} \leq R$$

$$\hookrightarrow x^2+y^2-2Ry+R^2+z^2 \leq R^2$$

$$\hookrightarrow x^2+y^2+z^2 \leq 2Ry$$

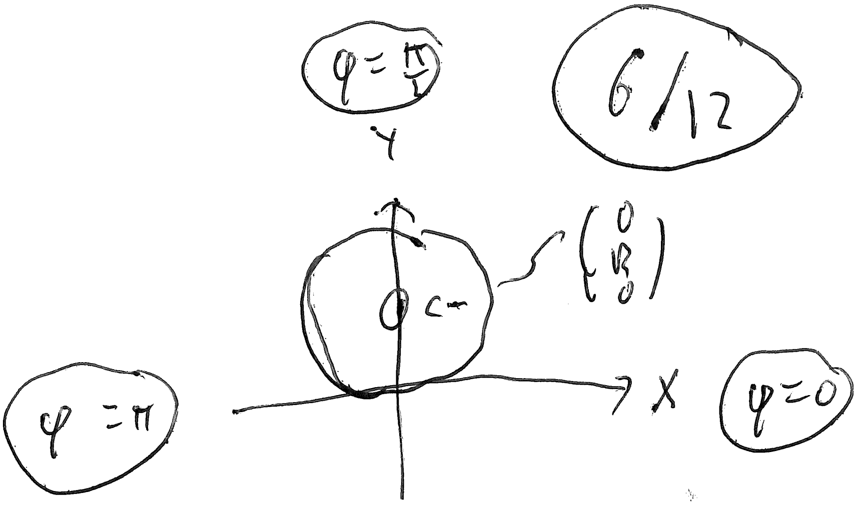
$$\hookrightarrow \rho^2 \leq 2R\cdot \rho \operatorname{sen}\theta \operatorname{sen}\varphi$$

$$\hookrightarrow \rho \leq 2R \operatorname{sen}\theta \operatorname{sen}\varphi$$

$$\hookrightarrow 0 \leq \rho \leq 2R \operatorname{sen}\theta \operatorname{sen}\varphi$$

$$0 \leq \theta \leq \pi$$

$6/12$



$0 \leq \varphi \leq \pi$

$\overline{Y} = R$ è evidente dal disegno

$$\iiint_{S_1} 1 \, dV = \frac{4\pi R^3}{3} \quad \text{poiché}$$

S_1 è una sfera di raggio R. (Si può anche calcolare con un integrale.)

$$\iiint_{S_1} y \, dV$$

$$= \int_{\varphi=0}^{\pi} \left(\int_{\theta=0}^{\pi} \left(\int_{r=0}^{2R \operatorname{sen}\theta \operatorname{sen}\varphi} (r \operatorname{sen}\theta \operatorname{sen}\varphi)(r^2 \operatorname{sen}\theta) \, dr \right) d\theta \right) d\varphi$$

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$$= \int_{\varphi=0}^{\pi} \left(\int_{\theta=0}^{\pi} \left(\int_{r=0}^{2R \operatorname{sen}\theta \operatorname{sen}\varphi} r^3 \operatorname{sen}^2\theta \operatorname{sen}\varphi \, dr \right) d\theta \right) d\varphi$$

$$= \int_{\varphi=0}^{\pi} \left(\int_{\theta=0}^{\pi} \frac{(2R \operatorname{sen}\theta \operatorname{sen}\varphi)^4}{4} \operatorname{sen}^2\theta \operatorname{sen}\varphi \, d\theta \right) d\varphi$$

$$= \frac{16R^4}{4} \int_{\varphi=0}^{\pi} \operatorname{sen}^5\varphi \left(\int_{\theta=0}^{\pi} \operatorname{sen}^6\theta \, d\theta \right) d\varphi$$

$$\Big[\int_{\theta=0}^{\pi} \operatorname{sen}^6\theta \, d\theta$$

$$= \cancel{-\frac{1}{6} \cos\theta \operatorname{sen}^5\theta \Big]_{\theta=0}^{\pi}}^{\,0} + \frac{5}{6} \int_{\theta=0}^{\pi} \operatorname{sen}^4\theta \, d\theta$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \int_{\theta=0}^{\pi} \operatorname{sen}^2\theta \, d\theta$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_{\theta=0}^{\pi} 1 \, d\theta = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \pi \Big]$$

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$$\left[\int_{\varphi=0}^{\pi} \operatorname{sen}^5 \varphi \, d\varphi \right.$$

$$= -\frac{1}{5} \cos\varphi \cancel{\operatorname{sen}^4 \varphi} \Big]_{\varphi=0}^{\pi} + \frac{4}{5} \int_{\varphi=0}^{\pi} \operatorname{sen}^3 \varphi \, d\varphi$$

$$= \frac{4}{5} \cdot \frac{2}{3} \cdot \int_{\varphi=0}^{\pi} \operatorname{sen} \varphi \, d\varphi = \frac{4}{5} \cdot \frac{2}{3} \cdot 2$$

$$\iiint_{S_1} y \, dV = \frac{16 R^4}{4} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 2 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \pi$$

$$= 4 \pi R^4 \cdot \frac{2}{6} = \frac{4 \pi R^4}{3}$$

$$\bar{y} = \frac{\iiint_{S_1} y \, dV}{\iiint_{S_1} 1 \, dV} = \frac{4 \pi R^4 / 3}{4 \pi R^3 / 3} = R$$

9/12

$\boxed{1}$

v	x	y	z	=
1	2	3	4	5
5	6	7	8	4
9	10	11	12	3
13	14	15	16	2
17	18	19	20	1
21	22	23	24	0

$E_2 \leftarrow E_2 - 5E_1$

$E_3 \leftarrow E_3 - 9E_1$

$E_4 \leftarrow E_4 - 13E_1$

$E_5 \leftarrow E_5 - 17E_1$

$E_6 \leftarrow E_6 - 21E_1$

1	2	3	4	5
0	-4	-8	-12	-21
0	-8	-16	-24	-42
0	-12	-24	-36	-63
0	-16	-32	-48	-84
0	-20	-40	-60	-105

10/12

$E2 \leftarrow \frac{-1}{4}E2 \longrightarrow$

$$\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
0 & 1 & 2 & 3 & 21/4 \\
0 & -8 & -16 & -24 & -42 \\
0 & -12 & -24 & -36 & -63 \\
0 & -16 & -32 & -48 & -84 \\
0 & -20 & -40 & -60 & -105
\end{array}$$

$E1 \leftarrow E1-2E2$

$E3 \leftarrow E3+8E2$

$E4 \leftarrow E4+12E2$

$E5 \leftarrow E5+16E2$

$E6 \leftarrow E6+20E2$

$$\begin{array}{ccccc}
v & x & y & z & = \\
1 & 0 & -1 & -2 & -11/2 \\
0 & 1 & 2 & 3 & 21/4 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}$$

$v - y - 2z = -11/2 \rightarrow v = y + 2z - 11/2$

$x + 2y + 3z = 21/4 \rightarrow x = -2y - 3z + 21/4$

11/12

$v = y + 2z - 11/2$

$x = -2y - 3z + 21/4$

$y = y$

$z = z$

$$\begin{pmatrix} v \\ x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -11/2 \\ 21/4 \\ 0 \\ 0 \end{pmatrix}$$

<u>Controllo</u>: $v + 2x + 3y + 4z$

$$\begin{aligned} = \ & (y + 2z - 11/2) \\ & + 2(-2y - 3z + 21/4) \\ & + 3(y \qquad\qquad) \\ & + 4(\quad z \qquad) \end{aligned} = 0y + 0z - \frac{11}{2} + \frac{21}{2} = \frac{10}{2}$$

$= 5$ ✓

12/12

$5v+6x+7y+8z$

$= 5(y+2z-11/2)$
$+6(-2y-3z+21/4)$
$+7(y)$
$+8(z)$

$= 0y+0z-\frac{110}{4}+\frac{126}{4}$

$= \frac{16}{4} = 4$ ✓

$9v+10x+11y+12z = 0y+0z-\frac{99}{2}+\frac{105}{2}$

$= 9(y+2z-11/2)$
$+10(-2y-3z+21/4)$
$+11(y)$
$+12(z)$

$= \frac{6}{2} = 3$ ✓