DA SCANNERIZZARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 15 luglio 2025

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

Le formule per le coordinate sferiche sono:

$$x = r\,\mathrm{sen}\,\theta\cos\phi \qquad y = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\,\mathrm{sen}\,\theta\,dr\,d\theta\,d\phi$$

A. Usare la sostituzione $u = \cos\theta$ e calcolare

$$\int \frac{\mathrm{sen}\,\theta}{\cos^2\theta - \mathrm{sen}^2\,\theta}\,d\theta$$

B. Sia $R > 0$ un parametro fisso. Usare le coordinate polari (r e θ) e calcolare la media di xy sul quarto del disco definito da:

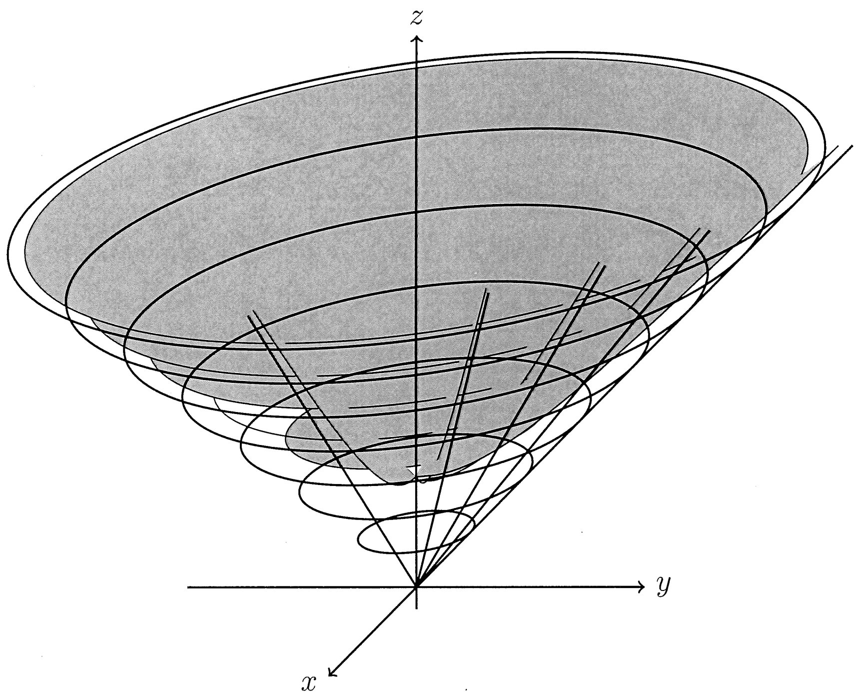
$$x, y \geq 0 \qquad\qquad x^2 + y^2 \leq R^2$$

C. Sia $\mathcal{F}$ la forma 3-dimensionale definita da:

$$z \geq 0 \qquad\qquad x^2 + y^2 \leq z^2 \leq x^2 + y^2 + 1$$

Si noti che $\mathcal{F}$ si estende fino all'infinito. Usare le coordinate sferiche e calcolare

$$\iiint_{\mathcal{F}} \frac{1}{x^2 + y^2 + z^2} \, dV$$



D. Siano

$$\begin{aligned} A &= x + y + z \\ B &= 3x + 2y + z \\ C &= 6x + 3y + z \end{aligned}$$

Usare il metodo di Gauss–Jordan e trovare le formule per x, y, e z in termini di A, B, e C. (Indicazione: Se il metodo è seguito senza errore e senza variazione, non saranno necessari denominatori.)

$\boxed{A}$

$u = \cos\theta$

$du = -\operatorname{sen}\theta \, d\theta$

$\operatorname{sen}^2\theta = 1 - \cos^2\theta = 1 - u^2$

$$\int \frac{\operatorname{sen}\theta}{\cos^2\theta - \operatorname{sen}^2\theta} \, d\theta = - \int \frac{1}{u^2 - (1-u^2)} \, du$$

$$= - \int \frac{1}{2u^2 - 1} \, du$$

$$\frac{1}{2u^2-1} = \frac{1}{(\sqrt{2}\cdot u - 1)} \cdot \frac{1}{(\sqrt{2}\cdot u + 1)}$$

$$= \frac{A}{\sqrt{2}\cdot u - 1} + \frac{B}{\sqrt{2}\cdot u + 1}$$

$$1 = A(\sqrt{2}\cdot u + 1) + B(\sqrt{2}\cdot u - 1)$$

$$= (A\sqrt{2} + B\sqrt{2})\,u + A - B$$

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$A\sqrt{2} + B\sqrt{2} = 0 \to A = -B$

$A - B = 1 \to -2B = 1$

$\to B = -1/2$

$A = +1/2$

$$-\int \frac{1}{2u^2-1}\,du = -\int \frac{1}{2}\frac{1}{\sqrt{2}\cdot u - 1} - \frac{1}{2}\frac{1}{\sqrt{2}\cdot u+1}\,du$$

$$= \frac{-1}{2}\frac{1}{\sqrt{2}}\log(\sqrt{2}\cdot u - 1) + \frac{1}{2}\frac{1}{\sqrt{2}}\log(\sqrt{2}\cdot u + 1)$$

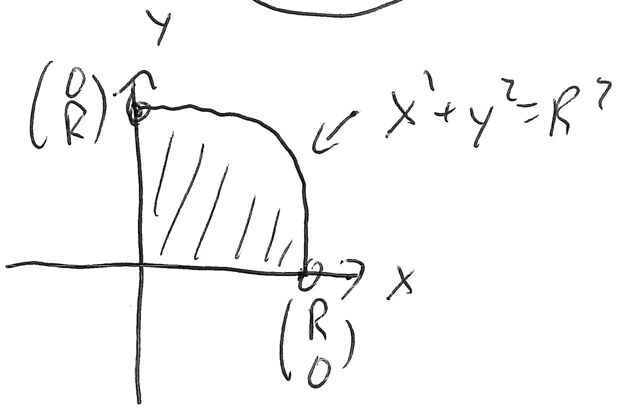
$$= \frac{1}{2}\frac{1}{\sqrt{2}}\left(\log(\sqrt{2}\cdot u + 1) - \log(\sqrt{2}\cdot u - 1)\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\left(\log(\sqrt{2}\cdot\cos\theta + 1) - \log(\sqrt{2}\cdot\cos\theta - 1)\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}\cos\theta + 1}{\sqrt{2}\cos\theta - 1}\right)$$

(3/9)

$\boxed{\text{U}}$



$$x^2+y^2 \le R^2 \to \rho^2 \le R^2 \to \rho \le R$$

$$0 \le \rho \le R$$

$$x, y \ge 0 \to \rho\cos\theta, \rho \operatorname{sen}\theta \ge 0$$

$$\to \cos\theta, \operatorname{sen}\theta \ge 0$$

$$\to 0 \le \theta \le \pi/2$$

$$\iint\limits_{\substack{x,y\ge 0 \\ x^2+y^2\le R^2}} 1\, dA = \int_{\theta=0}^{\pi/2} \left(\int_{\rho=0}^{R} \rho \, d\rho \right) d\theta$$

$$= \frac{\pi}{2} \cdot \frac{R^2}{2} = \frac{\pi R^2}{4} \quad \text{(giusto)}$$

$$\iint\limits_{\substack{x,y\ge 0 \\ x^2+y^2\le R^2}} xy\, dA = \int_{\theta=0}^{\pi/2} \left(\int_{\rho=0}^{R} (\rho\cos\theta)(\rho \operatorname{sen}\theta) \cdot \rho \, d\rho \right) d\theta$$

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$$= \int_{\theta=0}^{\pi/2} \cos\theta \cdot \operatorname{sen}\theta \left(\int_{r=0}^{R} r^3 \, dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos\theta \cdot \operatorname{sen}\theta \cdot \frac{R^4}{4} \, d\theta$$

$$= \left. \frac{\operatorname{sen}^2\theta}{2} \right]_{\theta=0}^{\pi/2} \cdot \frac{R^4}{4} = \frac{1}{2} \cdot \frac{R^4}{4} = \frac{R^4}{8}$$

$$\overline{xy} = \frac{\displaystyle\iint_{\substack{x,y \geq 0 \\ x^2+y^2 \leq R^2}} xy \, dA}{\displaystyle\iint_{\substack{x,y \geq 0 \\ x^2+y^2 \leq R^2}} 1 \, dA} = \frac{R^4/8}{\pi R^2/4}$$

$$= \frac{R^4}{8} \cdot \frac{4}{\pi R^2} = \frac{R^2}{2\pi}$$

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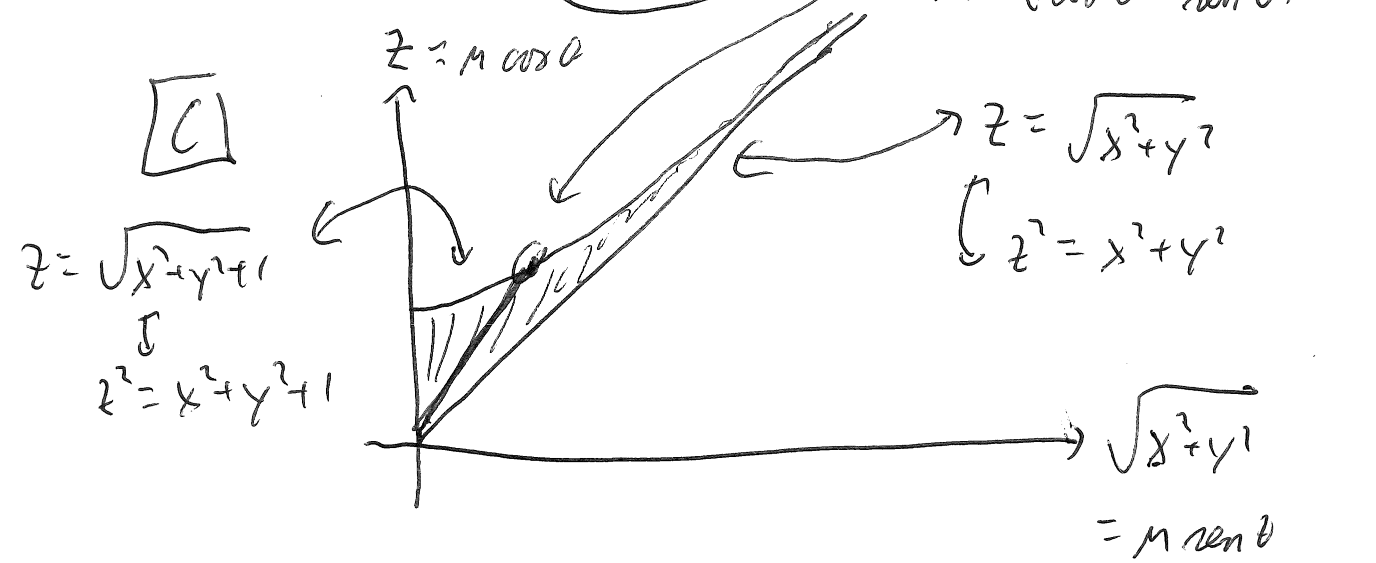
$\boxed{C}$



$$z^2 \geq x^2+y^2 \Leftrightarrow (\rho\cos\theta)^2 \geq (\rho\operatorname{sen}\theta)^2$$

$$\Leftrightarrow \cos^2\theta \geq \operatorname{sen}^2\theta$$

$$\Leftrightarrow \cos\theta \geq \operatorname{sen}\theta$$

$$\Leftrightarrow 1 \geq \operatorname{sen}\theta/\cos\theta = \operatorname{tg}\theta$$

$$\Leftrightarrow \pi/4 = \operatorname{arctg}(1) \geq \theta$$

$$0 \leq \theta \leq \pi/4$$

$$z^2 \leq x^2+y^2+1 \Leftrightarrow (\rho\cos\theta)^2 \leq (\rho\operatorname{sen}\theta)^2+1$$

$$\Leftrightarrow \rho^2(\cos^2\theta - \operatorname{sen}^2\theta) \leq 1$$

$$\Leftrightarrow \rho^2 \leq 1/(\cos^2\theta - \operatorname{sen}^2\theta)$$

$$\Leftrightarrow \rho \leq (\cos^2\theta - \operatorname{sen}^2\theta)^{-1/2}$$

$$0 \leq \rho \leq (\cos^2\theta - \operatorname{sen}^2\theta)^{-1/2}$$

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In coordinate sferiche: $x^2+y^2+z^2=r^2$.

$$\iiint_{\mathcal{J}} \frac{1}{x^2+y^2+z^2}\, dV$$

$$= \int_{\varphi=-\pi}^{\pi} \left(\int_{\theta=0}^{\pi/4} \left(\int_{r=0}^{(\cos^2\theta-\sin^2\theta)^{-1/2}} \frac{1}{r^2} r^2 \sin\theta \, dr \right) d\theta \right) d\varphi$$

$$= 2\pi \int_{\theta=0}^{\pi/4} \left(\int_{r=0}^{(\cos^2\theta-\sin^2\theta)^{-1/2}} \sin\theta \, dr \right) d\theta$$

$$= 2\pi \int_{\theta=0}^{\pi/4} \frac{\sin\theta}{\sqrt{\cos^2\theta-\sin^2\theta}} \, d\theta$$

$u=\cos\theta$

$du=-\sin\theta\, d\theta$

$\sin^2\theta=1-\cos^2\theta=1-u^2$

$\theta=0 \leftrightarrow u=\cos(0)=1$

$\theta=\pi/4 \leftrightarrow u=\cos(\pi/4)=\frac{1}{\sqrt{2}}$

$$= -2\pi \int_{u=1}^{1/\sqrt{2}} \frac{1}{\sqrt{u^2-(1-u^2)}}\, du$$

$$= 2\pi \int_{u=1/\sqrt{2}}^{1} \frac{1}{\sqrt{2u^2-1}}\, du$$

$u = \sec t / \sqrt{2}$

$du = (\tan t \sec t / \sqrt{2})\, dt$

$2u^2 - 1 = \sec^2 t - 1 = \tan^2 t$

$\sqrt{2u^2-1} = \tan t$

$u = \frac{1}{\sqrt{2}} \Leftrightarrow \frac{\sec t}{\sqrt{2}} = \frac{1}{\sqrt{2}} \Leftrightarrow \sec t = 1 \Leftrightarrow \cos t = 1$
$\Leftrightarrow t = 0$

$u = 1 \Leftrightarrow \frac{\sec t}{\sqrt{2}} = 1 \Leftrightarrow \sec t = \sqrt{2} \Leftrightarrow \cos t = \frac{1}{\sqrt{2}}$
$\Leftrightarrow t = \pi/4$

$$= 2\pi \int_{t=0}^{\pi/4} \frac{1}{\tan t} \cdot \frac{\tan t \sec t}{\sqrt{2}}\, dt$$

$$= \frac{2\pi}{\sqrt{2}} \int_{t=0}^{\pi/4} \sec t\, dt = \sqrt{2} \cdot \pi \cdot \log(\tan t + \sec t)\Big]_{t=0}^{\pi/4}$$

$$= \sqrt{2} \cdot \pi \left(\log(1+\sqrt{2}) - \log(0+1)\right)$$

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$= \sqrt{2}\cdot\pi\cdot\log(1+\sqrt{2})$

$\boxed{D}$

x	y	z	=	A	B	C
1	1	1		1	0	0
3	2	1		0	1	0
6	3	1		0	0	1

$\xrightarrow[E3 \leftarrow E3 - 6E1]{E2 \leftarrow E2 - 3E1}$

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 1 & 0 \\ 0 & -3 & -5 & -6 & 0 & 1 \end{array}$$

$\xrightarrow{E2 \leftarrow -E2}$

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & -3 & -5 & -6 & 0 & 1 \end{array}$$

~~E1 ← E1 + E2~~
~~E3 ← E3 + 2E2~~

$\xrightarrow[E3 \leftarrow E3 + 3E2]{E1 \leftarrow E1 - E2}$

$$\begin{array}{cccccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 3 & -3 & 1 \end{array}$$

$E1 \leftarrow E1+E3$

$\longrightarrow$

$E2 \leftarrow E2-2E3$

x	y	z	= A	B	C
1	0	0	1	−2	1
0	1	0	−3	5	−2
0	0	1	3	−3	1

$x = A - 2B + C$

$y = -3A + 5B - 2C$

$z = 3A - 3B + C$

<u>Controllo</u>:

$$x+y+z = (A-2B+C) + (-3A+5B-2C) + (3A-3B+C) = 1A + 0B + 0C = A \checkmark$$

$$3x+2y+z = 3(A-2B+C) + 2(-3A+5B-2C) + (3A-3B+C) = 0A + 1B + 0C = B \checkmark$$

$$6x+3y+z = 6(A-2B+C) + 3(-3A+5B-2C) + (3A-3B+C) = 0A + 0B + 1C = C \checkmark$$