DA SCANNERIZZARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 9 settembre 2025

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

Le formule per le coordinate cilindriche sono:

$$x = r\cos\theta \qquad y = r\,\mathrm{sen}\,\theta \qquad z = z \qquad\qquad dx\,dy\,dz = r\,dr\,d\theta\,dz$$

A. Usare la sostituzione $x = y^2$ e calcolare

$$\int \frac{y^5}{y^2(y^2+1)(y^2+4)}\,dy$$

B. Siano B e R parametri fissi con $0 < B < R$. Usare le coordinate polari (r e θ) e calcolare

$$\iint_{\substack{x^2+y^2\le R^2\\ y\ge B}} y\,dA$$

C. Siano B e R parametri fissi con $0 < B < R$. Usare le coordinate cilindriche (r, θ e z) e calcolare

$$\iiint_{\substack{x^2+y^2+z^2 \leq R^2 \\ z \geq B}} z\, dV$$

D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema:

$$\begin{aligned} v + x + y + z &= 4 \\ v + 2x + 3y + 4z &= 1 \\ v + 3x + 6y + 10z &= 0 \\ v + 4x + 10y + 20z &= 1 \\ v + 5x + 15y + 35z &= 4 \end{aligned}$$

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$\boxed{A}$

$$x = y^2$$

$$dx = 2y\,dy \rightsquigarrow \frac{1}{2}\,dx = y\,dy$$

$$\int \frac{y^5}{y^2(y^2+1)(y^2+4)}\,dy$$

$$= \frac{1}{2}\int \frac{(y^2)^2}{y^2(y^2+1)(y^2+4)}\,dx$$

$$= \frac{1}{2}\int \frac{x^2}{x(x+1)(x+4)}\,dx =$$

$$= \frac{1}{2}\int \frac{x}{(x+1)(x+4)}\,dx$$

$$\frac{x}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

$$\hookrightarrow \quad x = A(x+4) + B(x+1)$$

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$x=-1 \rightarrow -1 = A(-1+4) \rightarrow -1 = 3A$

$\rightarrow A = -1/3$

$x=-4 \rightarrow -4 = B(-4+1) \rightarrow -4 = -3B$

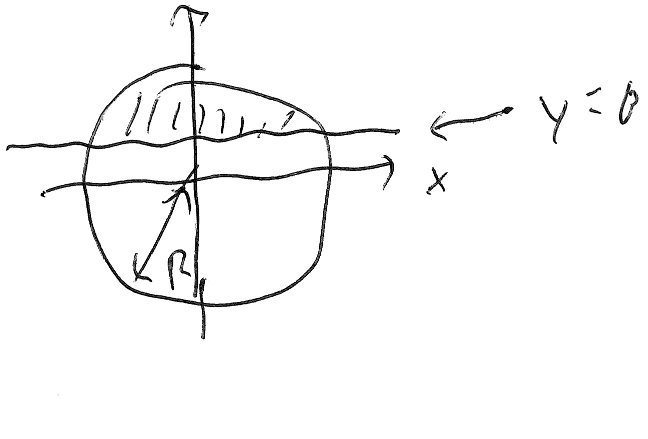
$\rightarrow B = 4/3$

$$\frac{1}{2}\int \frac{x}{(x+1)(x+4)}dx = \frac{1}{2}\int -\frac{1}{3}\frac{1}{x+1} + \frac{4}{3}\frac{1}{x+4}dx$$

$$= -\frac{1}{6}\log(x+1) + \frac{2}{3}\log(x+4)$$

$$= -\frac{1}{6}\log(y^2+1) + \frac{2}{3}\log(y^2+4)$$

[3]



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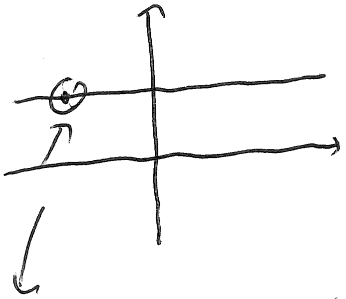


$\operatorname{sen}(\theta_0) = b/R$

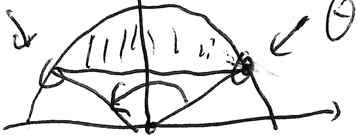
$y = b \leftrightarrow r \operatorname{sen}\theta = b$

$\leftrightarrow r = b/\operatorname{sen}\theta$

$\theta_0 \leq \theta \leq \pi - \theta_0$

$b/\operatorname{sen}\theta \leq r \leq R$

$\theta = \pi - \theta_0$



$\theta = \theta_0$

$$\int_{\theta=\theta_0}^{\pi-\theta_0} \left(\int_{r=b/\operatorname{sen}\theta}^{R} (r \operatorname{sen}\theta) \cdot r \, dr \right) d\theta$$

$$= \int_{\theta=\theta_0}^{\pi-\theta_0} \operatorname{sen}\theta \cdot \frac{1}{3} r^3 \Big]_{r=b/\operatorname{sen}\theta}^{R} d\theta$$

$$= \frac{1}{3} \int_{\theta=\theta_0}^{\pi-\theta_0} R^3 \operatorname{sen}\theta - b^3 \frac{\operatorname{sen}\theta}{\operatorname{sen}^3\theta} \, d\theta$$

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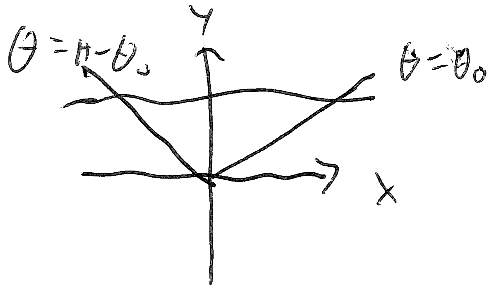
$$= \frac{1}{3}\left[\int_{\theta=\theta_0}^{\pi-\theta_0} R^3 \operatorname{sen}\theta - B^3 \csc^2\theta \, d\theta\right]$$

$$= \frac{1}{3}\left[-R^3\cos\theta\Big]_{\theta=\theta_0}^{\pi-\theta_0} + B^3 \operatorname{ctg}\theta\Big]_{\theta=\theta_0}^{\pi-\theta_0}\right]$$

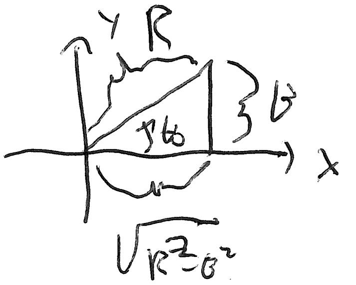


$\cos(\pi-\theta_0) = -\cos(\theta_0)$

$\operatorname{sen}(\pi-\theta_0) = \operatorname{sen}(\theta_0)$

$\operatorname{ctg}(\pi-\theta_0) = -\operatorname{ctg}(\theta_0)$

$$= \frac{2}{3}\left[R^3 \cos\theta_0 - B^3 \operatorname{ctg}\theta_0\right]$$



$\cos\theta_0 = \frac{AD}{HP} = \frac{\sqrt{R^2-B^2}}{R}$

$\operatorname{ctg}\theta_0 = \frac{AD}{OP} = \frac{\sqrt{R^2-B^2}}{B}$

$$= \frac{2}{3}\left[R^3\frac{\sqrt{R^2-B^2}}{R} - B^3\frac{\sqrt{R^2-B^2}}{B}\right]$$

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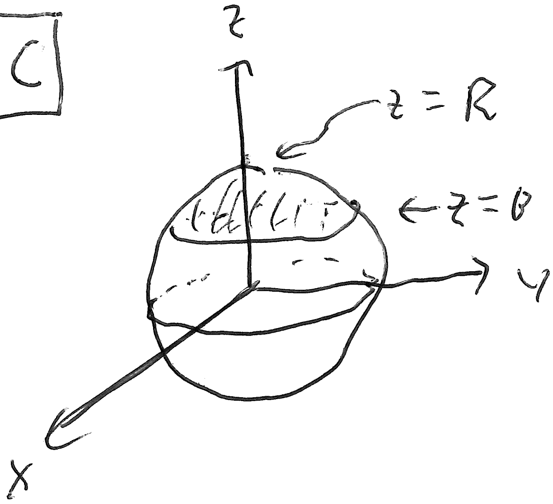
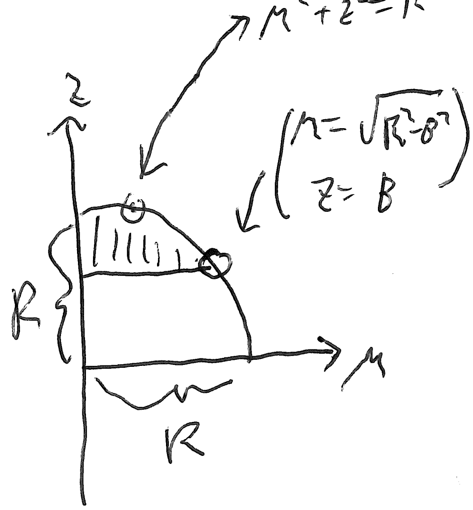
$$= \frac{2}{3}\sqrt{R^2-B^2}\left[R^2-B^2\right] = \frac{2}{3}(R^2-B^2)^{3/2}$$

$\boxed{C}$





$$\iiint\limits_{\substack{x^2+y^2+z^2\le R^2 \\ z\ge B}} z\,dV = \int_{\theta=-\pi}^{\pi}\left(\int_{r=0}^{\sqrt{R^2-B^2}}\left(\int_{z=B}^{\sqrt{R^2-r^2}} z\,r\,dz\right)dr\right)d\theta$$

$$= 2\pi\left(\int_{r=0}^{\sqrt{R^2-B^2}} r\cdot\frac{1}{2}z^2\Big]_{z=B}^{z=\sqrt{R^2-r^2}}\,dr\right)$$

$$= \pi\int_{r=0}^{\sqrt{R^2-B^2}} r\left(R^2-r^2-B^2\right)dr$$

$$= \pi\left(\left[\frac{r^2}{2}(R^2-B^2)-\frac{r^4}{4}\right]_{r=0}^{\sqrt{R^2-B^2}}\right)$$

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$$= \pi \left(\frac{1}{2} (R^2 - \theta^2)^2 - \frac{1}{4} (R^2 - \theta^2)^2 \right)$$

$$= \frac{\pi}{4} (R^2 - \theta^2)^4$$

$\boxed{D}$

V	X	Y	Z	=
1	1	1	1	4
1	2	3	4	1
1	3	6	10	0
1	4	10	20	1
1	5	15	35	4

E2 ← E2 − E1
E3 ← E3 − E1
E4 ← E4 − E1
E5 ← E5 − E1

1	1	1	1	4
0	1	2	3	−3
0	2	5	9	−4
0	3	9	19	−3
0	4	14	34	0

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$E1 \leftarrow E1 - E2$
$E3 \leftarrow E3 - 2E2$
$\longrightarrow$
$E4 \leftarrow E4 - 3E2$
$E5 \leftarrow E5 - 4E2$

$$\begin{matrix} 1 & 0 & -1 & -2 & 7 \\ 0 & 1 & 2 & 3 & -3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 3 & 10 & 6 \\ 0 & 0 & 6 & 22 & 12 \end{matrix}$$

$E1 \leftarrow E1 + E3$
$E2 \leftarrow E2 - 2E3$
$\longrightarrow$
$E3 \leftarrow E3 - 3E3$
$E4 \leftarrow E4 - 6E4$

$$\begin{matrix} 1 & 0 & 0 & 1 & 9 \\ 0 & 1 & 0 & -3 & -7 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{matrix}$$

$E1 \leftarrow E1 - E4$
$E2 \leftarrow E2 + 3E4$
$\longrightarrow$
$E3 \leftarrow E3 - 3E4$
$E5 \leftarrow E5 - 4E4$

$$\begin{array}{ccccc} V & X & Y & Z & = \\ \hline 1 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

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$v = 9$

$x = -7$

$y = 2$

$z = 0$

Controllo:

$v + x + y + z = 9 - 7 + 2 + 0 = 4$ v

$v + 2x + 3y + 4z$

$= 9 - 14 + 6 + 0 = 1$ v

$v + 3x + 6y + 10z$

$= 9 - 21 + 12 = 0$ v

$v + 4x + 10y + 20z$

$= 9 - 28 + 20 + 0 = 1$ v

$v + 5x + 15y + 35z$

$= 9 - 35 + 30 + 0 = 4$ v