DA SCANNERIZZARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 10 febbraio 2026

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

A. Calcolare

$$\int \frac{z^4}{(z+1)^4}\,dz$$

B. Siano A e R due parametri fissi con $0 < A < R$. Si consideri la zona $\mathcal{Z}$ del piano data da:

$$x^2 + y^2 \leq R^2 \qquad\qquad 0 \leq y \leq A$$

Fare un disegno di $\mathcal{Z}$. Poi usare le coordinate polari e calcolare l'area di $\mathcal{Z}$. (Indicazione: è necessario esprimere l'integrale come la somma di 3 pezzi.)

C.

1. Calcolare la media di p^2, ponderata dalla lunghezza, sul intervallo definito da $0 \leq p \leq 1$.
2. Calcolare la media di p^2, ponderata dall'area, sul triangolo definito da $0 \leq p \leq q \leq 1$.
3. Calcolare la media di p^2, ponderata dal volume, sul tetraedro definito da $0 \leq p \leq q \leq r \leq 1$.
4. Calcolare la media di p^2, ponderata dal 4-volume, sul "simplesso" definito da $0 \leq p \leq q \leq r \leq s \leq 1$.
5. *Senza fare nuovi calcoli*, tentare di indovinare la risposta che varrebbe in 5 dimensioni.

D. Usare il metodo di Gauss–Jordan e trovare la soluzione del sistema:

$$\begin{aligned} p+q+r+s &= A \\ p+q-r-s &= B \\ p-q+r-s &= C \\ p-q-r+s &= D \end{aligned}$$

per le incognite p, q, r, e s in termini di A, B, C, e D, da considerare come valori noti.

MMfor 1/11

$\boxed{A}$ $\deg(z^4)=4 \geq \deg(z+1)^4=4$

$$(z+1)^4 = z^4+4z^3+6z^2+4z+1$$

$$\begin{array}{r|l} z^4 & 1 \\ \hline -(z^4+4z^3+6z^2+4z+1) & z^4+4z^3+6z^2+4z+1 \\ \hline -4z^3-6z^2-4z-1 & \end{array}$$

$$\hookrightarrow z^4 = 1\cdot(z^4+4z^3+6z^2+4z+1)$$
$$+(-4z^3-6z^2-4z-1)$$

$$\hookrightarrow \frac{z^4}{(z+1)^4} = 1+\frac{(-4z^3 6z^2-4z-1)}{(z+1)^4}$$

$$\frac{-4z^3-6z^2-4z-1}{(z+1)^4} = \frac{A}{z+1}+\frac{B}{(z+1)^2}+\frac{C}{(z+1)^3}+\frac{D}{(z+1)^4}$$

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$$-4z^3-6z^2-4z-1 = A(z+1)^3+B(z+1)^2+C(z+1)+D$$

$$\begin{aligned} = {} & A(z^3+3z^2+3z+1) \\ & +B(z^2+2z+1) \\ & +C(z+1) \\ & +D(1) \end{aligned}$$

$$-4 = A$$

$$-6 = 3A+B = -12+B \rightsquigarrow B=6$$

$$-4 = 3A+2B+C = -12+12+C \rightsquigarrow C=-4$$

$$-1 = A+B+C+D = -4+6-4+D \rightsquigarrow D=1$$

$$\int \frac{z^4}{(z+1)^3}\,dz = \int 1 - \frac{4}{z+1} + \frac{6}{(z+1)^2} - \frac{4}{(z+1)^3} + \frac{1}{(z+1)^4}\,dz$$

$$= z - 4\log(z+1) - \frac{6}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{3}\frac{1}{(z+1)^3}$$

[illegible] 3/11

OPPURE:

$u = z+1$

$z = u-1$

$z^4 = u^4 - 4u^3 + 6u^2 - 4u + 1$

$du = dz$

$$\int \frac{z^4}{(z+1)^4}\, dz = \int \frac{u^4 - 4u^3 + 6u^2 - 4u + 1}{u^4}\, du$$

$$= \int 1 - \frac{4}{u} + \frac{6}{u^2} - \frac{4}{u^3} + \frac{1}{u^4}\, du$$

$$= z - 4 \log u - \frac{6}{u} + \frac{2}{u^2} - \frac{1}{3}\frac{1}{u^3}$$

$$= z+1 - 4 \log(z+1) - \frac{6}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{3}\frac{1}{(z+1)^3}$$

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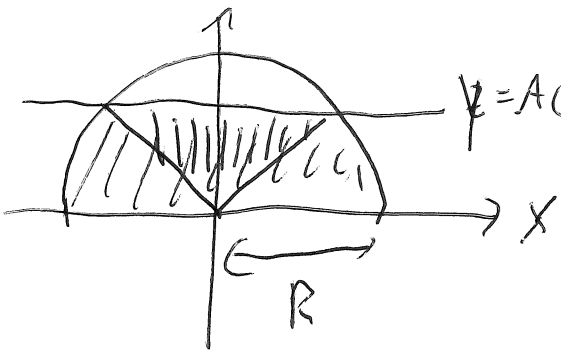
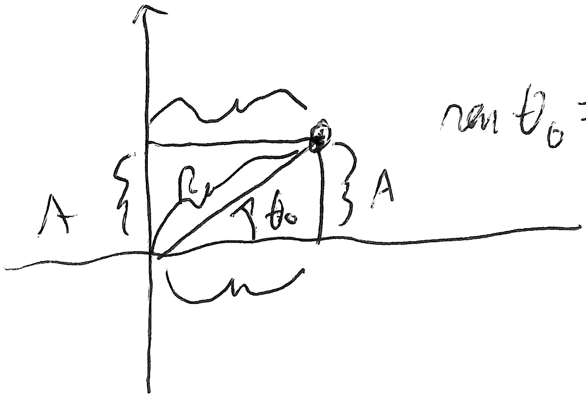
$\boxed{B}$



$y = A \longleftrightarrow r\,\mathrm{sen}\,\theta = A \longleftrightarrow r = \dfrac{A}{\mathrm{sen}\,\theta}$

$x^2 + y^2 \leq R^2$

$\hookrightarrow r \leq R$



$\mathrm{sen}\,\theta_0 = \dfrac{OP}{1P} = \dfrac{A}{R}$

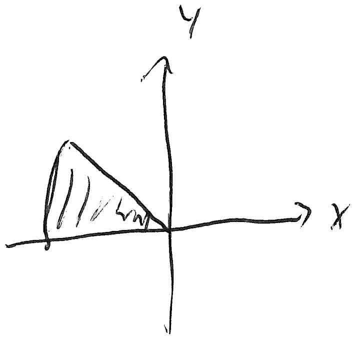
$y \leq A$

$\hookrightarrow r\,\mathrm{sen}\,\theta \leq A$

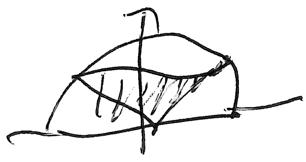
$\hookrightarrow r \leq \dfrac{A}{\mathrm{sen}\,\theta}$

$$\text{area}(Z) = \iint_Z 1 \, dA$$

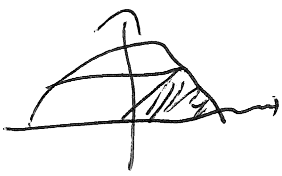
$$= \int_{\pi-\theta_0}^{\pi} \left(\int_{r=0}^{R} 1\, r \, dr \right) d\theta$$



$$+ \int_{\theta_0}^{\pi-\theta_0} \left(\int_{r=0}^{A/\mathrm{sen}\,\theta} 1 \cdot r \, dr \right) d\theta$$



$$+ \int_{0}^{\theta_0} \left(\int_{r=0}^{R} 1 \cdot r \, dr \right) d\theta$$



5/11 [illegible]

$$= \int_{\pi-\theta_0}^{\pi} \frac{R^2}{2}\, d\theta$$

$$+ \int_{\theta_0}^{\pi-\theta_0} \frac{1}{2}\left(\frac{A}{\sin\theta}\right)^2 d\theta$$

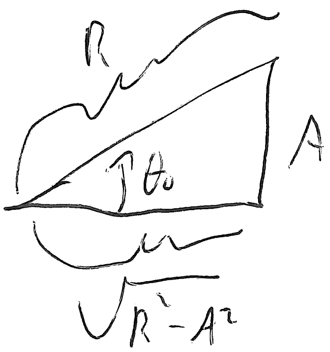
$$+ \int_{0}^{\theta_0} \frac{R^2}{2}\, d\theta$$

$$= \frac{R^2}{2}\theta_0 + \int_{\theta_0}^{\pi-\theta_0} \frac{1}{2} A^2 \csc^2\theta\, d\theta + \frac{R^2}{2}\theta_0$$

$$= R^2\theta_0 + \frac{1}{2}A^2 \int_{\theta_0}^{\pi-\theta_0} \csc^2\theta\, d\theta$$

$$= R^2\theta_0 + \frac{1}{2}A^2 \left(-\operatorname{ctg}\theta\right)\Big|_{\theta_0}^{\pi-\theta_0}$$

[



$$\operatorname{ctg}\theta_0 = \frac{AO}{OP} = \frac{\sqrt{R^2-A^2}}{A}$$

$$\operatorname{ctg}(\pi-\theta_0) = \frac{\cos(\pi-\theta_0)}{\sin(\pi-\theta_0)} = \frac{-\cos(\theta_0)}{\sin(\theta_0)} = \frac{-\sqrt{R^2-A^2}}{A}$$]

$$\iint_{\mathcal{Z}} 1\, dA$$

$$= R^2\theta_0 + \frac{1}{2}A^2\left(\frac{\sqrt{R^2-A^2}}{A} - \left(-\frac{\sqrt{R^2-A^2}}{A}\right)\right)$$

$$= R^2\theta_0 + A\sqrt{R^2-A^2}$$

con $\operatorname{sen}\theta_0 = \frac{A}{R}$, $\theta_0 = \operatorname{arcsen}\left(\frac{A}{R}\right)$

$\boxed{C}$ 1 $\int_{p=0}^{1} p^2\, dp \Big/ \int_{p=0}^{1} 1\, dp = \frac{1/3}{1} = \frac{1}{3}$

2 $\int_{q=0}^{1}\left(\int_{p=0}^{q} p^2\, dp\right) dq \Big/ \int_{q=0}^{1}\left(\int_{p=0}^{q} 1\, dp\right) dq$

$$= \int_{q=0}^{1} \frac{q^3}{3}\, dq \Big/ \int_{q=0}^{1} q\, dq = \frac{1/12}{1/2} = \frac{1}{6}$$

3 $$\frac{\int_{u=0}^{1}\left(\int_{q=0}^{u}\left(\int_{p=0}^{q} p^2\, dp\right) dq\right) du}{\int_{u=0}^{1}\left(\int_{q=0}^{u}\left(\int_{p=0}^{q} 1\, dp\right) dq\right) du}$$

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$$= \frac{\int_{u=0}^{1}\left(\int_{q=0}^{u} \frac{q^3}{3}\,dq\right)du}{\int_{u=0}^{1}\left(\int_{q=0}^{u} q\,dq\right)du} = \frac{\int_{u=0}^{1}\frac{u^4}{12}\,du}{\int_{u=0}^{1}\frac{u^2}{2}\,du}$$

$$= \frac{1/60}{1/6} = \frac{1}{10}$$

(4) $$\frac{\int_{s=0}^{1}\left(\int_{u=0}^{s}\left(\int_{q=0}^{u}\left(\int_{p=0}^{q} p^2\,dp\right)dq\right)du\right)ds}{\int_{s=0}^{1}\left(\int_{u=0}^{s}\left(\int_{q=0}^{u}\left(\int_{p=0}^{q} 1\,dp\right)dq\right)du\right)ds}$$

$$= \frac{\int_{s=0}^{1}\left(\int_{u=0}^{s}\left(\int_{q=0}^{u}\frac{q^3}{3}\,dq\right)du\right)ds}{\int_{s=0}^{1}\left(\int_{u=0}^{s}\left(\int_{q=0}^{u} q\,dq\right)du\right)ds}$$

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$$= \frac{\int_{s=0}^{1} \left(\int_{n=0}^{s} \frac{n^4}{12}\, dn \right) ds}{\int_{s=0}^{1} \int_{n=0}^{s} \frac{n^2}{2}\, dn)\, ds} = \frac{\int_{s=0}^{1} \frac{s^5}{60}\, ds}{\int_{s=0}^{1} \frac{s^3}{6}\, ds}$$

$$= \frac{1/360}{1/24} = \frac{1}{15}$$

5) $\frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}$

$+3 \quad +4 \quad +5 \quad +6$

D)

$$\begin{array}{cccc|cccc}
p & q & n & s = & A & B & C & D \\
\hline
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 1
\end{array}$$

$E2 \leftarrow E2 - E1$
$E3 \leftarrow E3 - E1$
$E4 \leftarrow E4 - E1$

$$\begin{array}{cccc|cccc}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \\
0 & -2 & 0 & -2 & -1 & 0 & 1 & 0 \\
0 & -2 & -2 & 0 & -1 & 0 & 0 & 1
\end{array}$$

~~[illegible]~~ 9/11

$$\xrightarrow{E_2 \leftrightarrow E_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{E_2 \leftarrow -\frac{1}{2}E_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \\ 0 & -2 & -2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[E_4 \leftarrow E_4 - 2E_2]{E_1 \leftarrow E_1 - E_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & -2 & -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{E_3 \leftarrow -\frac{1}{2}E_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & -2 & +2 & 0 & 0 & -1 & 1 \end{array}\right]$$

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$E_1 \leftarrow E_1 - E_3$

$\longrightarrow$

$E_4 \leftarrow E_4 + 2E_3$

$$\begin{array}{cccc|cccc}
1 & 0 & 0 & -1 & 0 & 1/2 & 1/2 & 0 \\
0 & 1 & 0 & 1 & 1/2 & 0 & -1/2 & 0 \\
0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\
0 & 0 & 0 & 4 & 1 & -1 & -1 & 1
\end{array}$$

$E_4 \leftarrow \frac{1}{4} E_4$

$$\begin{array}{cccc|cccc}
1 & 0 & 0 & -1 & 0 & 1/2 & 1/2 & 0 \\
0 & 1 & 0 & 1 & 1/2 & 0 & -1/2 & 0 \\
0 & 0 & 1 & 1 & 1/2 & -1/2 & 0 & 0 \\
0 & 0 & 0 & 1 & 1/4 & -1/4 & -1/4 & 1/4
\end{array}$$

$E_1 \leftarrow E_1 + E_4$

$E_2 \leftarrow E_2 - E_4$

$E_3 \leftarrow E_3 - E_4$

$$\begin{array}{cccc|cccc}
p & q & r & s & = A & B & C & D \\
\hline
1 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\
0 & 1 & 0 & 0 & 1/4 & 1/4 & -1/4 & -1/4 \\
0 & 0 & 1 & 0 & 1/4 & -1/4 & 1/4 & -1/4 \\
0 & 0 & 0 & 1 & 1/4 & -1/4 & -1/4 & 1/4
\end{array}$$

$$p = \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D$$

$$q = \frac{1}{4}A + \frac{1}{4}B - \frac{1}{4}C - \frac{1}{4}D$$

$$r = \frac{1}{4}A - \frac{1}{4}B + \frac{1}{4}C - \frac{1}{4}D$$

$$s = \frac{1}{4}A - \frac{1}{4}B - \frac{1}{4}C + \frac{1}{4}D$$

Controlle:

$$p + q + r + s = 1A + 0B + 0C + 0D = A$$

$$p + q - r - s = 0A + 1B + 0C + 0D = B$$

$$p - q + r - s = 0A + 0B + 1C + 0D = C$$

$$p - q - r + s = 0A + 0B + 0C + 1D = D$$