DA FOTOCOPIARE

Matematica 2, Laurea Triennale (Steger)

Prova Scritta di 31 marzo 2023

Imponiamo alcune *regole fiscali* affinché, in coscienza, si possa dare al candidato una buona votazione globale sulla base della prova scritta, anche quando i risultati dell'orale siano discutibili.

- La prova si affronta senza i libri e *senza le calcolatrici.* È permesso un formulario di una pagina (A4), ambedue lati, scritto a mano dallo stesso candidato.
- L'esame verrà svolto esclusivamente sui fogli messi a disposizione dal docente.

La durata della prova è di 3 ore, dalle ore 10.00 alle ore 13.00. La prova si concluderà puntualmente.

I compiti corretti saranno a disposizione mercoledì 6 giugno, alle 12.30, al 1° piano del palazzo didattico di via Vienna.

Le formule per le coordinate polari sono:

$$x = r\cos\theta \qquad y = r \operatorname{sen}\theta \qquad\qquad dx\,dy = r\,dr\,d\theta$$

Le formule per le coordinate sferiche sono:

$$x = r\operatorname{sen}\theta\cos\phi \qquad y = r\operatorname{sen}\theta\operatorname{sen}\phi \qquad z = r\cos\theta$$
$$dx\,dy\,dz = r^2\operatorname{sen}\theta\,dr\,d\theta\,d\phi$$

A. Usare la sostituzione $u = t^2$ e calcolare

$$\int \frac{t^5}{(t^2-1)(t^2-4)}\,dt$$

B. Siano $C, D > 0$ parametri fissi. Si consideri il triangolo $\mathcal{T}''$ con vertici $\binom{0}{0}$, $\binom{0}{D}$, e $\binom{C}{D}$. Disegnare $\mathcal{T}''$ e poi usare *le coordinate polari* per il calcolo della media di x^2 su $\mathcal{T}''$.

C. Sia $R > 0$ un parametro fisso. Sia $\mathcal{F}$ il volume definito da:

$$\begin{gathered} x^2 + y^2 + z^2 \leq R^2 \\ 0 \leq z \leq x \end{gathered}$$

$\mathcal{F}$ è una fetta che rappresenta l'ottava parte di una sfera. Calcolare il volume di $\mathcal{F}$ con *le coordinate sferiche.*

D. Usare il metodo di Gauss–Jordan e trovare una parametrizzazione dello spazio delle soluzioni del sistema:

$$\begin{array}{cccccl} a & +2b & +3c & +4d & +5e & = 6 \\ a & +3b & +5c & +7d & +9e & = 11 \\ a & +4b & +7c & +10d & +13e & = 16 \\ a & +5b & +9c & +13d & +17e & = 21 \\ a & +6b & +11c & +16d & +21e & = 26 \\ a & +7b & +13c & +19d & +25e & = 31 \\ a & +8b & +15c & +22d & +29e & = 36 \end{array}$$

1/13

A

$$u = t^2$$

$$du = 2t\, dt$$

$$\int \frac{t^5}{(t^2-1)(t^2-4)}\, dt$$

$$= \frac{1}{2} \int \frac{t^4}{(t^2-1)(t^2-4)} \cdot 2t\, dt$$

$$= \frac{1}{2} \int \frac{u^2}{(u-1)(u-4)}\, du$$

$$(u-1)(u-4) = u^2 - 5u + 4$$

$$\begin{array}{r|l} u^2 & u^2-5u+4 \\ \hline -(u^2-5u+4) & 1 \\ \hline 5u-4 & \end{array}$$

2/13

$$u^2 = 1 \cdot (u^2 - 5u + 4) + (5u - 4)$$

$$\frac{u^2}{u^2 - 5u + 4} = 1 + \frac{5u-4}{u^2-5u+4}$$

$$\frac{5u-4}{(u-1)(u-4)} = \frac{A}{u-1} + \frac{B}{u-4}$$

$$(5u-4) = A(u-4) + B(u-1)$$

$$u = 4 \rightsquigarrow (5 \cdot 4 - 4) = 0 + B(4-1)$$

$$\rightarrow 16 = 3B \rightarrow B = 16/3$$

$$u = 1 \rightsquigarrow (5 \cdot 1 - 4) = A(1-4) + 0$$

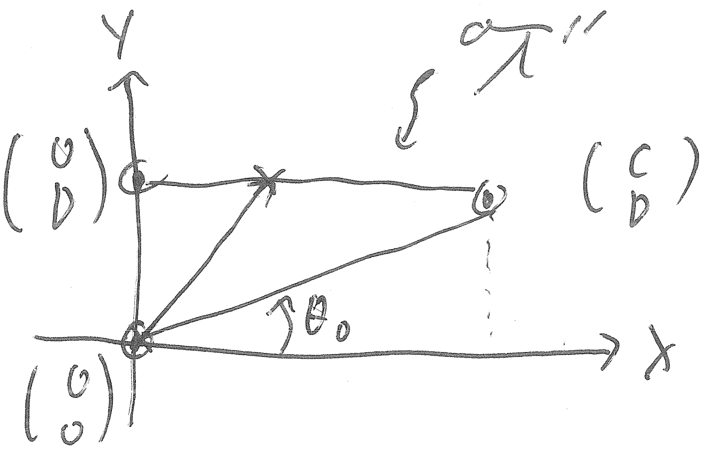
$$\rightarrow 1 = -3A \rightsquigarrow A = -1/3$$

$$\int \frac{u^2}{(u-1)(u-4)}\,du = \int 1 - \frac{1}{3}\frac{1}{u-1} + \frac{16}{3}\frac{1}{u-4}\,du$$

$$= \frac{1}{2}\left(u - \frac{1}{3}\log(u-1) + \frac{16}{3}\log(u-4)\right)$$

$$= \frac{1}{2}\left(t^2 - \frac{1}{3}\log(t^2-1) + \frac{16}{3}\log(t^2-4)\right)$$

B



Al soffitto di π''

$$D = y = r \,\mathrm{sen}\,\theta$$

onde $\quad r = D/\mathrm{sen}\,\theta$

$$\iint_{\pi''} I \, dA = \int_{\theta=\theta_0}^{\pi/2} \left(\int_{r=0}^{D/\mathrm{sen}\,\theta} I \cdot r \, dr \right) d\theta$$

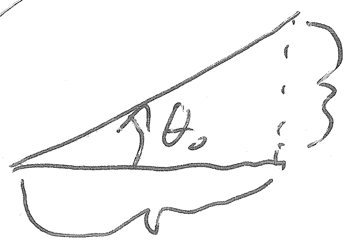
$$= \int_{\theta=\theta_0}^{\pi/2} \frac{1}{2}\left(D/\mathrm{sen}\,\theta\right)^2 d\theta$$

4/13

$$= \frac{1}{2} D^2 \int_{\theta_0}^{\pi/2} \csc^2\theta \, d\theta$$

$$= -\frac{1}{2} D^2 \operatorname{ctg}\theta \Big]_{\theta=\theta_0}^{\pi/2}$$

$$= -\frac{1}{2} D^2 (0 - \operatorname{ctg}\theta_0)$$



$OP = D \rightsquigarrow \operatorname{ctg}\theta_0 = \frac{AD}{OP}$

$AD = C$ $\qquad = \frac{C}{D}$

$$= \frac{1}{2} D^2 \frac{C}{D} = \frac{1}{2} CD$$

$$(= \text{area}(^{o}\mathcal{T}''))$$

$$\iint_{^{o}\mathcal{T}''} x^2 \, dA = \int_{\theta=\theta_0}^{\pi/2} \left(\int_{r=0}^{D/\operatorname{sen}\theta} (r\cos\theta)^2 \cdot r \, dr \right) d\theta$$

5/13

$$= \int_{\theta=\theta_0}^{\pi/2} \cos^2\theta \left(\int_{r=0}^{D/\operatorname{sen}\theta} r^3\, dr \right) d\theta$$

$$= \int_{\theta=\theta_0}^{\pi/2} \cos^2\theta \cdot \frac{1}{4} \left(\frac{D}{\operatorname{sen}\theta}\right)^4 d\theta$$

$$= \frac{1}{4} D^4 \int_{\theta=\theta_0}^{\pi/2} \cos^2\theta \cdot \frac{1}{\operatorname{sen}^4\theta}\, d\theta$$

$$= \frac{1}{4} D^4 \int_{\theta=\theta_0}^{\pi/2} (1-\operatorname{sen}^2\theta) \frac{1}{\operatorname{sen}^4\theta}\, d\theta$$

$$= \frac{1}{4} D^4 \int_{\theta=\theta_0}^{\pi/2} \csc^4\theta - \csc^2\theta\, d\theta$$

$$\left[\int_{\theta=\theta_0}^{\pi/2} \csc^2\theta\, d\theta = -\operatorname{ctg}\theta \right]_{\theta=\theta_0}^{\pi/2}$$

$$= C/D$$

$$\int_{\theta=\theta_0}^{\pi/2} \csc^4\theta \, d\theta$$

$$= -\frac{1}{3}\operatorname{ctg}\theta\csc^2\theta\Big]_{\theta=\theta_0}^{\pi/2} + \frac{2}{3}\int_{\theta=\theta_0}^{\pi/2}\csc^2\theta\, d\theta$$

$$= 0 + \frac{1}{3}\operatorname{ctg}\theta_0\csc^2\theta_0 + \frac{2}{3}\frac{C}{D}$$

$$= \frac{1}{3}\operatorname{ctg}\theta_0(1+\operatorname{ctg}^2\theta_0) + \frac{2}{3}\frac{C}{D}$$

$$= \frac{1}{3}\frac{C}{D}\left(1+\frac{C^2}{D^2}\right) + \frac{2}{3}\frac{C}{D}$$

$$= \frac{C}{D} + \frac{1}{3}\frac{C^3}{D^3}$$

]

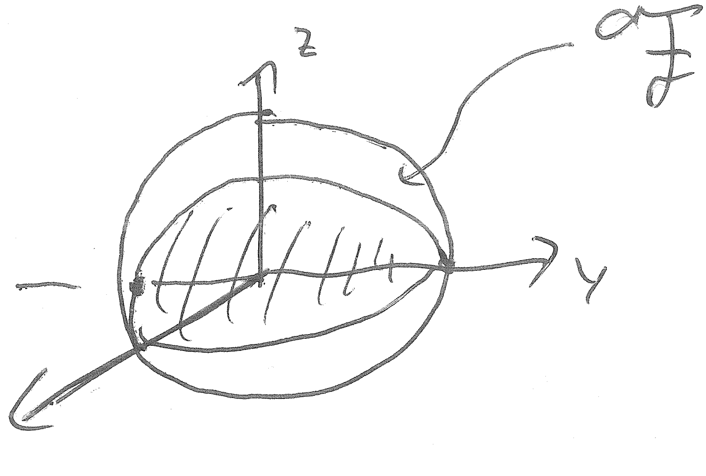
$$\iint_{\pi''} x^2\, dA = \frac{1}{4}D^4\left(\frac{C}{D} + \frac{1}{3}\frac{C^3}{D^3} - \frac{C}{D}\right)$$

$$= \frac{1}{4}D^4\left(\frac{1}{3}\frac{C^3}{D^3}\right) = \frac{1}{12}C^3D$$

$$\overline{x^2} = \frac{1}{12}C^3D \Big/ \frac{1}{2}CD = \boxed{\frac{1}{6}C^2}$$

7/13

$\boxed{C}$



$$x^2+y^2+z^2 \leq R^2 \to r^2 \leq R^2$$
$$\to r \leq R$$

e $\quad 0 \leq r \quad$ sempre.

$$0 \leq z \leq x \to 0 \leq r\cos\theta \leq r \operatorname{sen}\theta \cos\varphi$$
$$\to 0 \leq \cos\theta \leq \operatorname{sen}\theta \cos\varphi$$

$$0 \leq \cos\theta \to \theta \leq \frac{\pi}{2}$$

$$\cos\theta \leq \operatorname{sen}\theta\cos\varphi \to \frac{1}{\cos\varphi} \leq \frac{\operatorname{sen}\theta}{\cos\theta} = \operatorname{tg}\theta$$
$$\to \operatorname{arctg}(1/\cos\varphi) \leq \theta$$

Risolvere $0 \le \operatorname{sen}\theta \cos\varphi$

$\rightsquigarrow 0 \le \cos\varphi$ (poiché $0 \le \theta \le \pi$, $\operatorname{sen}\theta \ge 0$)

$\rightsquigarrow -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$$\iiint_{\mathcal{F}} 1 \, dV$$

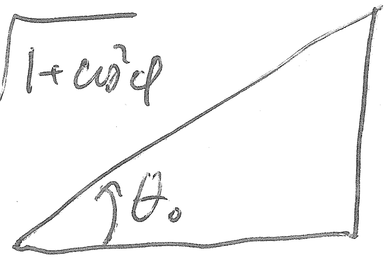
$$= \int_{\varphi=-\pi/2}^{\pi/2} \left(\int_{\theta=\operatorname{arctg}(1/\cos\varphi)}^{\pi/2} \left(\int_{r=0}^{R} r^2 \operatorname{sen}\theta \, dr \right) d\theta \right) d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=-\pi/2}^{\pi/2} \left(\int_{\theta=\operatorname{arctg}(1/\cos\varphi)}^{\pi/2} \operatorname{sen}\theta \, d\theta \right) d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=-\pi/2}^{\pi/2} -\cos\theta \Big]_{\theta=\operatorname{arctg}(1/\cos\varphi)}^{\pi/2} d\varphi$$

9/13

$IP = \sqrt{1+\cos^2\varphi}$



$OP = 1$

$AI = \cos\varphi$

$\theta_0 = \operatorname{arctg}\left(\frac{1}{\cos\varphi}\right)$

$\operatorname{tg}\theta_0 = \frac{OP}{AI} = \frac{1}{\cos\varphi}$

$\cos\theta_0 = \frac{AI}{IP} = \frac{\cos\varphi}{\sqrt{1+\cos^2\varphi}}$

$$\iiint_{\mathcal{F}} 1\, dV = \frac{R^3}{3} \int_{\varphi=-\pi/2}^{+\pi/2} (0 + \cos\theta_0)\, d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=-\pi/2}^{\pi/2} \frac{\cos\varphi}{\sqrt{1+\cos^2\varphi}}\, d\varphi$$

$$= \frac{R^3}{3} \int_{\varphi=-\pi/2}^{\pi/2} \frac{\cos\varphi}{\sqrt{1+1-\operatorname{sen}^2\varphi}}\, d\varphi$$

$u = \operatorname{sen}\varphi \qquad du = \cos\varphi\, d\varphi$

$\operatorname{sen}(\pm\pi/2) = \pm 1$

$$= \frac{R^3}{3} \int_{u=-1}^{+1} \frac{1}{\sqrt{2-u^2}}\, du$$

10/13

$u = \sqrt{2}\,\text{sen}\,t$

$du = \sqrt{2}\cos t\,dt$

$u = \pm 1 \rightarrow \text{sen}\,t = \pm 1/\sqrt{2}$

$\rightarrow t = \pm \pi/4$

$$= \frac{R^3}{3} \int_{t=-\pi/4}^{+\pi/4} \frac{1}{\sqrt{2 - 2\,\text{sen}^2 t}} \sqrt{2}\cos t\,dt$$

$$= \frac{R^3}{3} \int_{t=-\pi/4}^{\pi/4} \frac{1}{\sqrt{2}\sqrt{1-\text{sen}^2 t}} \sqrt{2}\cos t\,dt$$

$$= \frac{R^3}{3} \int_{t=-\pi/4}^{\pi/4} 1\,dt = \frac{R^3}{3} \cdot \frac{\pi}{2}$$

$$= \frac{\pi R^3}{6} \qquad \left[= \frac{1}{8} \frac{4\pi R^3}{3} \right]$$

11/13

D

a	b	c	d	e	=
1	2	3	4	5	6
1	3	5	7	9	11
1	4	7	10	13	16
1	5	9	13	17	21
1	6	11	16	21	26
1	7	13	19	25	31
1	8	15	22	29	36

$E2 \leftarrow E2 - E1$, $E3 \leftarrow E3 - E1$, $E4 \leftarrow E4 - E1$,
$E5 \leftarrow E5 - E1$, $E6 \leftarrow E6 - E1$, $E7 \leftarrow E7 - E1$

1	2	3	4	5	6
0	1	2	3	4	5
0	2	4	6	8	10
0	3	6	9	12	15
0	4	8	12	16	20
0	5	10	15	20	25
0	6	12	18	24	30

$E1 \leftarrow E1 - 2E2$
$E3 \leftarrow E3 - 2E2$
$E4 \leftarrow E4 - 3E2$
$E5 \leftarrow E5 - 4E2$
$E6 \leftarrow E6 - 5E2$
$E7 \leftarrow E7 - 6E2$

12/13

a	b	c	d	e	=
1	0	-1	-2	-3	-4
0	1	2	3	4	5
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

$a - c - 2d - 3e = -4$

$b + 2c + 3d + 4e = 5$

$0 = 0$

$0 = 0$

$0 = 0$

$0 = 0$

$0 = 0$

13/13

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} c + 2d + 3e - 4 \\ -2c - 3d - 4e + 5 \\ c \\ d \\ e \end{pmatrix}$$

Controllo:

$a + 2b + 3c + 4d + 5e$

$$\begin{aligned} &= (c + 2d + 3e - 4) \\ &+ 2(-2c - 3d - 4e + 5) \\ &+ 3(c) \\ &+ 4(d) \\ &+ 5(e) \end{aligned}$$

$= 0c + 0d + 0e + 6 = 6$ ✓

eccetera.